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1. Answer the following true/false questions:
   (a) If $T : V \to V$ has $T^* = T$ then all of the eigenvalues of T are real.
   (b) If V is a finite-dimensional inner product space and $T : V \to V$ is Hermitian, then there exists an orthonormal basis of V consisting of eigenvectors for T.
   (c) Real symmetric matrices are diagonalizable.
   (d) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \text{tr}(AB)$, then Φ is a bilinear form on V.
   (e) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \det(AB)$, then Φ is a bilinear form on V.
   (f) Every bilinear form on $\mathbb{R}^n$ is diagonalizable.
   (g) Every quadratic form on $\mathbb{R}^n$ is diagonalizable.
   (h) Congruent matrices have the same eigenvalues.
   (i) The critical point $(0, 0, 0)$ of $f(x, y, z) = x^2 + y^2 - z^2$ is a saddle point.
   (j) The graph of $x^2 - 4xy + 9y^2 = 1$ is a hyperbola.
   (k) If $A \in M_{n\times n}(\mathbb{C})$, the singular values of A are the eigenvalues of A^*A.
   (l) Every matrix $A \in M_{n\times n}(\mathbb{C})$ can be written as a product $A = PDQ$ where P and Q are unitary and D is diagonal.

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2. Suppose V is an inner product space and $T : V \to V$ is linear and has an adjoint T^*.
   (a) If T is Hermitian, show that $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$.
   (b) If $T(\mathbf{v})$ is orthogonal to $\mathbf{v}$ for all $\mathbf{v} \in V$, show that T must be the zero transformation. [Hint: Apply the property to $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and $\mathbf{w} = \mathbf{x} + i\mathbf{y}$, then take $\mathbf{y} = T\mathbf{x}$.]
   (c) Suppose $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$. Show that T is Hermitian.

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3. Suppose V is an inner product space and $T : V \to V$ is linear. Recall that we say T is an <u>isometry</u> if $\langle T(\mathbf{v}), T(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$; equivalently, T^*T is the identity on V.
   (a) If T is an isometry, show that any eigenvalue $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.
   (b) Conversely, suppose that V possesses an orthonormal basis of eigenvectors for T each of whose eigenvalues λ has $|\lambda| = 1$. Prove that T is an isometry. [Hint: Compute $\langle T(\mathbf{v}), T(\mathbf{w})\rangle$ in terms of this basis.]
   (c) If T is an isometry that is also Hermitian, show that T^2 is the identity map.

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4. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$.
   (a) Find the eigenvalues of A and a basis for each eigenspace.
   (b) Find a formula for the nth power A^n.
   (c) Solve the system of differential equations $y_1' = 2y_1 + 3y_2$, $y_2' = 3y_1 + 10y_2$ using any method.
   (d) Find an invertible rational matrix Q such that Q^TAQ is diagonal.
   (e) Describe the shape of $2x^2 + 6xy + 10y^2 = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.
   (f) Classify the critical point of $2x^2 + 6xy + 10y^2$ at $(0, 0)$ as a local min, local max, or saddle point.

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5. Suppose Φ is a symmetric bilinear form on a complex vector space V such that $\Phi(\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v}$, with equality only when $\mathbf{v} = \mathbf{0}$. Prove that V must be the zero vector space.

   Remark: This problem illustrates why we need the requirement of conjugate-symmetry, rather than regular symmetry, to have an interesting theory of inner products on complex vector spaces: the only "inner product" on a complex vector space with regular symmetry is the trivial pairing on the zero space!

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6. Suppose $A \in M_{n\times n}(\mathbb{C})$. Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$. [Hint: Put A in Jordan form.]

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7. Suppose $A \in M_{m\times n}(\mathbb{C})$. Show that $I_n + A^*A$ is positive definite. [Hint: Show A^*A is positive-semidefinite.]

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8. Let A, B be elements of $M_{n\times n}(F)$ for an arbitrary field F.
   (a) If A and B are similar over F, prove that A^T and B^T are also similar over F.
   (b) If A and B are congruent over F, prove that A^T and B^T are also congruent over F.
   (c) If A and B are similar over F and invertible, prove that A^{-1} and B^{-1} are also similar over F.

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9. An $n \times n$ real matrix A is called a Gram matrix if there exists an $n \times n$ real matrix B with $A = B^T B$.
   (a) Show that every Gram matrix is symmetric and positive semidefinite. [Hint: If $\mathbf{v}$ is an eigenvector, consider $\langle A\mathbf{v}, \mathbf{v}\rangle$ under the standard dot product.]
   (b) Conversely, suppose that A is a positive-semidefinite symmetric matrix. Show that A is a Gram matrix.

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10. Suppose that A is a symmetric positive-semidefinite matrix. Prove that the singular values of A are the eigenvalues of A.

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11. Suppose $A \in M_{n\times n}(F)$ is congruent to a diagonal matrix. Prove that A must be symmetric.

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12. Let V be a finite-dimensional inner product space. Recall that we say $T : V \to V$ is an isometry when T^*T is the identity map on V. Prove that T is an isometry if and only if T is invertible and all of the singular values of T equal 1.

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13. Suppose V and W are finite-dimensional inner product spaces and $T : V \to W$ has nonzero singular values $\sigma_1 \geq \cdots \geq \sigma_r$ and orthonormal singular value bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \ldots, \mathbf{w}_m\}$ with $T(\mathbf{v}_i) = \sigma_i \mathbf{w}_i$ for $1 \leq i \leq r$ and $T(\mathbf{v}_i) = \mathbf{0}$ for $i > r$.
    (a) Show that for any $\mathbf{v} \in V$, $T(\mathbf{v}) = \sigma_1 \langle \mathbf{v}, \mathbf{v}_1\rangle \mathbf{w}_1 + \cdots + \sigma_r \langle \mathbf{v}, \mathbf{v}_r\rangle \mathbf{w}_r$.
    (b) Show that for any $\mathbf{v} \in V$, $||T(\mathbf{v})|| \leq \sigma_1 ||\mathbf{v}||$ and that equality can occur.
    (c) Show that for any $\mathbf{w} \in W$, $T^*(\mathbf{w}) = \sigma_1 \langle \mathbf{w}, \mathbf{w}_1\rangle \mathbf{v}_1 + \cdots + \sigma_r \langle \mathbf{w}, \mathbf{w}_r\rangle \mathbf{v}_r$.
