

1. (a) True: If $T\mathbf{v} = \lambda\mathbf{v}$ then $\lambda\langle\mathbf{v}, \mathbf{v}\rangle = \langle T\mathbf{v}, \mathbf{v}\rangle = \langle\mathbf{v}, T\mathbf{v}\rangle = \bar{\lambda}\langle\mathbf{v}, \mathbf{v}\rangle$ so $\lambda = \bar{\lambda}$.
 - (b) True: This is the spectral theorem for Hermitian operators.
 - (c) True: This is the spectral theorem applied to real operators.
 - (d) True: Φ is linear in both A and B , so it is a bilinear form on V .
 - (e) False: here $\Phi(A, B) = \det(A)\det(B)$, but the determinant is not a linear function on matrices.
 - (f) False: bilinear forms on any field of characteristic not 2 are diagonalizable if and only if they are symmetric.
 - (g) True: this follows from the spectral theorem, since the associated matrix is symmetric.
 - (h) False: with $Q = 2I_2$ we see A and $Q^T A Q = 4A$ are congruent, but the eigenvalues of $4A$ are four times the eigenvalues of A .
 - (i) True: the Hessian matrix is diagonal with diagonal entries 1, 1, -1 hence it has a positive and negative eigenvalue, so the point is a saddle point.
 - (j) False: the associated quadratic form is positive definite so the graph is an ellipse.
 - (k) False: the singular values are the square roots of the eigenvalues of A^*A .
 - (l) True: in the singular value decomposition $A = U\Sigma V^*$, the matrices U and V are unitary (so V^* is also unitary) and Σ is diagonal.
-

2. (a) If $T^* = T$ then $\langle T\mathbf{v}, \mathbf{v}\rangle = \langle\mathbf{v}, T^*\mathbf{v}\rangle = \langle\mathbf{v}, T\mathbf{v}\rangle = \overline{\langle T\mathbf{v}, \mathbf{v}\rangle}$ so $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real.
 - (b) With $\mathbf{v} = \mathbf{x} + \mathbf{y}$ we have $0 = \langle T\mathbf{v}, \mathbf{v}\rangle = \langle T\mathbf{x} + T\mathbf{y}, \mathbf{x} + \mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{x}\rangle + \langle T\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, \mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, \mathbf{x}\rangle$. Likewise with $\mathbf{w} = \mathbf{x} + i\mathbf{y}$ we have $0 = \langle T\mathbf{w}, \mathbf{w}\rangle = \langle T\mathbf{x} + iT\mathbf{y}, \mathbf{x} + i\mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{x}\rangle - i\langle T\mathbf{x}, \mathbf{y}\rangle + i\langle T\mathbf{y}, \mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle = -i\langle T\mathbf{x}, \mathbf{y}\rangle + i\langle T\mathbf{y}, \mathbf{x}\rangle$. In order for both of these to be true we must have $\langle T\mathbf{x}, \mathbf{y}\rangle = \langle T\mathbf{y}, \mathbf{x}\rangle = 0$ for all \mathbf{x}, \mathbf{y} . But now setting $\mathbf{y} = T\mathbf{x}$ we see $\langle T\mathbf{x}, T\mathbf{x}\rangle = 0$ hence $T\mathbf{x} = \mathbf{0}$, so T is the zero transformation.
 - (c) If $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$ then similarly to (a) we see $\langle T\mathbf{v}, \mathbf{v}\rangle = \overline{\langle T\mathbf{v}, \mathbf{v}\rangle} = \overline{\langle\mathbf{v}, T^*\mathbf{v}\rangle} = \langle T^*\mathbf{v}, \mathbf{v}\rangle$, so $\langle(T - T^*)\mathbf{v}, \mathbf{v}\rangle = 0$ for all \mathbf{v} . Applying part (b) to $T - T^*$ this means $T - T^*$ is the zero transformation, so $T = T^*$ hence T is Hermitian.
-

3. (a) Suppose $T\mathbf{v} = \lambda\mathbf{v}$. Then $\langle\mathbf{v}, \mathbf{v}\rangle = \langle T\mathbf{v}, T\mathbf{v}\rangle = \langle\lambda\mathbf{v}, \lambda\mathbf{v}\rangle = \lambda\bar{\lambda}\langle\mathbf{v}, \mathbf{v}\rangle = |\lambda|^2\langle\mathbf{v}, \mathbf{v}\rangle$. Since $\langle\mathbf{v}, \mathbf{v}\rangle > 0$, cancelling gives $|\lambda|^2 = 1$ so $|\lambda| = 1$.
 - (b) Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of eigenvectors for T with eigenvalues $\lambda_1, \dots, \lambda_n$ of absolute value 1. For any $\mathbf{v} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$ and $\mathbf{w} = b_1\mathbf{e}_1 + \dots + b_n\mathbf{e}_n$ by expanding and using orthonormality, we have $\langle\mathbf{v}, \mathbf{w}\rangle = \sum_{i,j} \langle a_i\mathbf{e}_i, b_j\mathbf{e}_j\rangle = \sum_{i,j} a_i b_j \langle\mathbf{e}_i, \mathbf{e}_j\rangle = a_1 b_1 + \dots + a_n b_n$ and then $\langle T\mathbf{v}, T\mathbf{w}\rangle = \sum_{i,j} \langle a_i\lambda_i\mathbf{e}_i, b_j\lambda_j\mathbf{e}_j\rangle = \sum_{i,j} a_i\lambda_i b_j\lambda_j \langle\mathbf{e}_i, \mathbf{e}_j\rangle = a_1\lambda_1 b_1\lambda_1 + \dots + a_n\lambda_n b_n\lambda_n = a_1 b_1 + \dots + a_n b_n = \langle\mathbf{v}, \mathbf{w}\rangle$.
 - (c) Since T is an isometry, T^*T is the identity, but since T is Hermitian, $T^* = T$, hence T^2 is the identity.
-

4. (a) Eigenvalues are 1, 11 with eigenspaces spanned by $(-3, 1)$ and $(1, 3)$ respectively.
 - (b) Taking $Q = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ with $Q^{-1} = Q^T$ gives $A^n = Q \begin{bmatrix} 1 & 0 \\ 0 & 11^n \end{bmatrix} Q^T = \frac{1}{10} \begin{bmatrix} 9 + 11^n & 3(11^n - 1) \\ 3(11^n - 1) & 1 + 9 \cdot 11^n \end{bmatrix}$.
 - (c) From the eigenvalue method the solutions are $(y_1, y_2) = C_1(-3, 1)e^x + C_2(1, 3)e^{11x}$.
 - (d) $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - 3/2R_1} \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 11/2 & -3/2 & 1 \end{array} \right]$ so we can take $Q = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}$.
 - (e) Since the eigenvalues are both positive, the quadratic form is positive definite and the conic is an ellipse.
 - (f) Since the eigenvalues are both positive, the critical point is a local minimum.
-

5. For any \mathbf{v} we have $\langle i\mathbf{v}, i\mathbf{v}\rangle \geq 0$. But $\langle i\mathbf{v}, i\mathbf{v}\rangle = i\langle\mathbf{v}, i\mathbf{v}\rangle = i\langle i\mathbf{v}, \mathbf{v}\rangle = -\langle\mathbf{v}, \mathbf{v}\rangle$ and this quantity is ≤ 0 because $\langle\mathbf{v}, \mathbf{v}\rangle \geq 0$. Thus we must have $\langle i\mathbf{v}, i\mathbf{v}\rangle = 0$ hence $i\mathbf{v} = \mathbf{0}$ hence $\mathbf{v} = \mathbf{0}$. Thus, V is the zero vector space.
-

6. Let J be the Jordan form of A with $J = Q^{-1}AQ$. As in class, $e^A = Q^{-1}e^JQ$, so $\det(e^A) = \det(Q^{-1}e^JQ) = \det(e^J)$. From the formula for the exponential of a Jordan block, or because J is upper-triangular, or by the spectral mapping theorem, the diagonal entries of e^J are just the exponentials of the diagonal entries of J ; namely, $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}$. Hence $\det(e^J) = e^{\lambda_1}e^{\lambda_2} \dots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \dots + \lambda_n} = e^{\text{tr}(A)}$, as claimed.

7. First note $I_n + A^*A$ is Hermitian since $(I_n + A^*A)^* = I_n^* + A^*(A^*)^* = I_n + A^*A$. Now suppose $A^*A\mathbf{v} = \lambda\mathbf{v}$: then $\|A\mathbf{v}\|^2 = \langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \mathbf{v}, A^*A\mathbf{v} \rangle = \langle \mathbf{v}, \lambda\mathbf{v} \rangle = \bar{\lambda} \langle \mathbf{v}, \mathbf{v} \rangle = \bar{\lambda} \|\mathbf{v}\|^2$ where the inner product is the standard one on \mathbb{C}^n : thus $\bar{\lambda} = \|T(\mathbf{v})\|^2 / \|\mathbf{v}\|^2 \geq 0$ and thus λ is a nonnegative real number. Then the eigenvalues of $I + A^*A$ are 1 plus the eigenvalues of A^*A hence they are all positive.

8. (a) Suppose $B = Q^{-1}AQ$. Then $B^T = (Q^{-1}AQ)^T = Q^T A^T (Q^{-1})^T = Q^T A^T (Q^T)^{-1} = R^{-1}A^T R$ where $R = (Q^T)^{-1}$. Thus, A^T and B^T are similar.
 (b) Suppose $B = Q^T A Q$. Then $B^T = (Q^T A Q)^T = Q^T A^T (Q^T)^T = Q^T A Q$.
 (c) Suppose $B = Q^{-1}AQ$. Then $B^{-1} = (Q^{-1}AQ)^{-1} = Q^{-1}A^{-1}(Q^{-1})^{-1} = Q^{-1}A^{-1}Q$.

9. (a) If $A = B^T B$ then $A^T = (B^T B)^T = B^T B = A$ so A is symmetric. If \mathbf{v} is an eigenvector of A with $A\mathbf{v} = \lambda\mathbf{v}$ then under the standard dot product we have $\lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle \lambda\mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle = \langle B^T B\mathbf{v}, \mathbf{v} \rangle = \langle B\mathbf{v}, B\mathbf{v} \rangle$ since $B^* = B^T$ as B is real. Then $\lambda = \langle B\mathbf{v}, B\mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and so A is positive semidefinite as its eigenvalues are nonnegative.

(b) If A is symmetric then by the spectral theorem, A is diagonalizable via an orthonormal change of basis, meaning there exists an orthogonal matrix Q (with $Q^T Q = I_n$) such that $Q A Q^{-1} = D$ is diagonal. Since the diagonal entries of D are the eigenvalues of A , which are nonnegative by the positive-semidefinite hypothesis, letting E be the matrix whose diagonal entries are the square roots of the diagonal entries of D , we have $E^T E = D$. Then $A = Q^{-1} D Q = Q^T E^T E Q = (E Q)^T (E Q) = B^T B$ for $B = E Q$, so A is a Gram matrix.

10. By definition the singular values are the square roots of the eigenvalues of $A^*A = A^2$, but by the spectral mapping theorem these are simply the squares of the eigenvalues of A . But since A is positive-semidefinite, each eigenvalue λ of A is nonnegative, hence the singular value $\sqrt{\lambda^2} = \lambda$.

11. If A is congruent to a diagonal matrix then there exists an invertible matrix Q such that $A = Q^T D Q$. Then $A^T = (Q^T D Q)^T = Q^T D^T (Q^T)^T = Q^T D Q = A$ since $D^T = D$ as D is diagonal: thus, A is symmetric.

12. If T is invertible and all of its singular values equal 1, then the eigenvalues of the Hermitian operator T^*T are the squares of the singular values, which are all 1. Since this operator is diagonalizable, this means the diagonalization is the identity, but this is equivalent to saying T^*T is the identity. Conversely, if T is an isometry, then T^*T is the identity on V so its singular values are the square roots of the eigenvalues of the identity map, so they are all 1.

13. (a) If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ then $\langle \mathbf{v}, \mathbf{v}_i \rangle = a_i$ since the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal. Then $T(\mathbf{v}) = a_1T(\mathbf{v}_1) + \dots + a_nT(\mathbf{v}_n) = a_1(\sigma\mathbf{w}_1) + \dots + a_r(\sigma\mathbf{w}_r) + \mathbf{0} + \dots + \mathbf{0} = \sigma_1 \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{w}_1 + \dots + \sigma_r \langle \mathbf{v}, \mathbf{v}_r \rangle \mathbf{w}_r$.

(b) If $\mathbf{v} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ then since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal we have $\|\mathbf{v}\|^2 = a_1^2 + \dots + a_n^2$. Then by (a) since $T(\mathbf{v}) = \sigma_1 a_1 \mathbf{w}_1 + \dots + \sigma_r a_r \mathbf{w}_r$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ is orthonormal we have $\|T(\mathbf{v})\|^2 = \sigma_1^2 a_1^2 + \dots + \sigma_r^2 a_r^2 \leq \sigma_1^2 (a_1^2 + \dots + a_r^2) \leq \sigma_1^2 (a_1^2 + \dots + a_n^2) = \sigma_1^2 \|\mathbf{v}\|^2$. Taking the square root yields the desired inequality, and equality occurs for $\mathbf{v} = \mathbf{v}_1$ since $\|\mathbf{v}\| = 1$ while $\|T\mathbf{v}\| = \sigma_1$.

(c) If $\mathbf{w} = b_1\mathbf{w}_1 + \dots + b_n\mathbf{w}_n$ then since $T^*(\mathbf{w}_i) = \sigma_i\mathbf{v}_i$ (as follows by applying T^* to both sides of $T(\mathbf{v}_i) = \sigma_i\mathbf{w}_i$), the formula follows by the same calculation as in (a).
