E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2026 ~ Practice Final A Answers

1. (a) False: need at least 4 vectors to span a 4-dimensional space.
   (b) True: for instance, a basis works.
   (c) True: this is the definition of one-to-one.
   (d) False: bases are in the wrong order for composition.
   (e) False: should be $\overline{\langle \mathbf{v}, \mathbf{w} \rangle} = \langle \mathbf{w}, \mathbf{v} \rangle$.
   (f) False: $(S + iT)^* = S^* - iT^*$.
   (g) True: $\ker(A) = 0$ iff the 0-eigenspace is 0.
   (h) True: Cayley-Hamilton applies to all matrices.
   (i) True: this follows from the spectral theorem.
   (j) False: bilinear forms on any field of characteristic not 2 are diagonalizable if and only if they are symmetric.
   (k) True: the Hessian matrix is diagonal with diagonal entries 1, 1, -1 hence it has a positive and negative eigenvalue, so the point is a saddle point.
   (l) False: the singular values are the square roots of the eigenvalues of A^*A.

2. If A is orthogonal then $A^{-1} = A^T$ so transposing gives $(A^{-1})^T = A = (A^{-1})^{-1}$ so A^{-1} is orthogonal.

3. By the nullity-rank theorem, we have $\dim(\operatorname{im} T) \leq \dim(\ker T) + \dim(\operatorname{im} T) = \dim V < \dim W$ so $\operatorname{im}(T)$ cannot equal W hence T is not onto.

4. (a) If $\ker(T) = \ker(T^2)$ let $\mathbf{w} \in \operatorname{im}(T) \cap \ker(T)$. Then $\mathbf{w} = T(\mathbf{v})$ so then $\mathbf{0} = T(\mathbf{w}) = T^2(\mathbf{v})$: thus $\mathbf{v} \in \ker(T^2)$ hence $\mathbf{v} \in \ker(T)$ and thus $\mathbf{w} = T(\mathbf{v}) = \mathbf{0}$.
   (b) If $\operatorname{im}(T) = \operatorname{im}(T^2)$ then $\dim(V) - \dim(\operatorname{im}(T)) = \dim(V) - \operatorname{im}(T^2)$ hence by nullity-rank this yields $\dim(\ker(T)) = \dim(\ker(T^2))$. But since $\ker(T) \subseteq \ker(T^2)$, since $T(\mathbf{v}) = \mathbf{0}$ implies $T^2(\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$, and these spaces are finite-dimensional, we see $\ker(T) = \ker(T^2)$. Then by (a), $\operatorname{im}(T) \cap \ker(T) = \{\mathbf{0}\}$.
   (c) The derivative map is onto so $\operatorname{im}(D) = \operatorname{im}(D^2) = \mathbb{R}[x]$, but nonzero constants are also in $\operatorname{im}(D) \cap \ker(D)$.

5. Observe that the pairing $\langle (a, b), (c, d) \rangle = ac + ad + bc + 3bd$ is an inner product on $\mathbb{R}^2$: [I1] and [I2] are direct calculations and for [I3] we have $\langle (a, b), (a, b) \rangle = a^2 + 2ab + 3b^2 = (a + b)^2 + 2b^2 \geq 0$ with equality only for $a = b = 0$. The desired inequality is just the triangle inequality for $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$: $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.

6. (a) $||\mathbf{u}_i + \mathbf{u}_j||^2 = \langle \mathbf{u}_i + \mathbf{u}_j, \mathbf{u}_i + \mathbf{u}_j \rangle = \langle \mathbf{u}_i, \mathbf{u}_i \rangle + \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \langle \mathbf{u}_j, \mathbf{u}_i \rangle + \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 1 + 0 + 0 + 1 = 2$.
   (b) $2 = ||\mathbf{u}_i + \mathbf{u}_j||^2 = \langle \mathbf{u}_i, \mathbf{u}_i \rangle + \langle \mathbf{u}_i, \mathbf{u}_j \rangle + \langle \mathbf{u}_j, \mathbf{u}_i \rangle + \langle \mathbf{u}_j, \mathbf{u}_j \rangle = 2 + 2 \langle \mathbf{u}_i, \mathbf{u}_j \rangle$ so $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$.

7. If $\lambda_1 = \lambda_2$ then $\mathbf{v}_1 + \mathbf{v}_2$ is in the λ_1-eigenspace. Conversely, since $T(\mathbf{v}_1 + \mathbf{v}_2) = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$ if $T(\mathbf{v}_1 + \mathbf{v}_2) = \mu(\mathbf{v}_1 + \mathbf{v}_2)$ then $(\lambda_1 - \mu)\mathbf{v}_1 + (\lambda_2 - \mu)\mathbf{v}_2 = \mathbf{0}$. If $\mu = \lambda_1 = \lambda_2$ then this relation holds trivially; otherwise at least one coefficient is nonzero so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent, and this can only occur when $\lambda_1 = \lambda_2$, so either way $\lambda_1 = \lambda_2$.

8. With $\beta = \{1, x, x^2, x^3\}$ we see $[D]_\beta^\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is upper-triangular, so the eigenvalues are $\lambda = 0, 0, 0, 0$ and the 0-eigenspace is 1-dimensional, so D is not diagonalizable. Its Jordan form must have one block with eigenvalue 0, so the Jordan form is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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9. (a) Suppose $T\mathbf{v} = \lambda\mathbf{v}$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle = \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle = \lambda\overline{\lambda} \langle \mathbf{v}, \mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle$. Since $\langle \mathbf{v}, \mathbf{v} \rangle > 0$, cancelling gives $|\lambda|^2 = 1$ so $|\lambda| = 1$.

   (b) Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of eigenvectors for T with eigenvalues $\lambda_1, \dots, \lambda_n$ of absolute value 1. For any $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ and $\mathbf{w} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n$ by expanding and using orthonormality, we have $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i,j} \langle a_i\mathbf{e}_i, b_j\mathbf{e}_j \rangle = \sum_{i,j} a_i\overline{b_j} \langle \mathbf{e}_i, \mathbf{e}_j \rangle = a_1\overline{b_1} + \cdots + a_n\overline{b_n}$ and then $\langle T\mathbf{v}, T\mathbf{w} \rangle = \sum_{i,j} \langle a_i\lambda_i\mathbf{e}_i, b_j\lambda_j\mathbf{e}_j \rangle = \sum_{i,j} a_i\lambda_i\overline{b_j\lambda_j} \langle \mathbf{e}_i, \mathbf{e}_j \rangle = a_1\lambda_1\overline{b_1\lambda_1} + \cdots + a_n\lambda_n\overline{b_n\lambda_n} = a_1\overline{b_1} + \cdots + a_n\overline{b_n} = \langle \mathbf{v}, \mathbf{w} \rangle$.

   (c) Since T is an isometry, T^*T is the identity, but since T is Hermitian, $T^* = T$, hence T^2 is the identity.

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10. (a) Suppose $B = Q^{-1}AQ$. Then $B^T = (Q^{-1}AQ)^T = Q^TA^T(Q^{-1})^T = Q^TA^T(Q^T)^{-1} = R^{-1}A^TR$ where $R = (Q^T)^{-1}$. Thus, A^T and B^T are similar.

    (b) Suppose $B = Q^TAQ$. Then $B^T = (Q^TAQ)^T = Q^TA^T(Q^T)^T = Q^TAQ$.

    (c) Suppose $B = Q^{-1}AQ$. Then $B^{-1} = (Q^{-1}AQ)^{-1} = Q^{-1}A^{-1}(Q^{-1})^{-1} = Q^{-1}A^{-1}Q$.

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11. By definition the singular values are the square roots of the eigenvalues of $A^*A = A^2$, but by the spectral mapping theorem these are simply the squares of the eigenvalues of A. But since A is positive-semidefinite, each eigenvalue λ of A is nonnegative, hence the singular value $\sqrt{\lambda^2} = \lambda$.

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12. (a) If $A = B^TB$ then $A^T = (B^TB)^T = B^TB = A$ so A is symmetric. If $\mathbf{v}$ is an eigenvector of A with $A\mathbf{v} = \lambda\mathbf{v}$ then under the standard dot product we have $\lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle \lambda\mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle = \left\langle B^TB\mathbf{v}, \mathbf{v} \right\rangle = \langle B\mathbf{v}, B\mathbf{v} \rangle$ since $B^* = B^T$ as B is real. Then $\lambda = \langle B\mathbf{v}, B\mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and so A is positive semidefinite as its eigenvalues are nonnegative.

    (b) If A is symmetric then by the spectral theorem, A is diagonalizable via an orthonormal change of basis, meaning there exists an orthogonal matrix Q (with $Q^TQ = I_n$) such that $QAQ^{-1} = D$ is diagonal. Since the diagonal entries of D are the eigenvalues of A, which are nonnegative by the positive-semidefinite hypothesis, letting E be the matrix whose diagonal entries are the square roots of the diagonal entries of D, we have $E^TE = D$. Then $A = Q^{-1}DQ = Q^TE^TEQ = (EQ)^T(EQ) = B^TB$ for $B = EQ$, so A is a Gram matrix.

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1. (a) False: can't have > 2 independent vectors in a 2-dimensional space.
   (b) False: the set of all vectors in V is not linearly independent.
   (c) True: the condition says that T is a bijection.
   (d) False: this is only true if W is finite-dimensional.
   (e) True: this is the Cauchy-Schwarz inequality.
   (f) False: for instance, a 2×2 Jordan λ-block and λI_2 have eigenvalues λ, λ but are not similar.
   (g) False: the 1-eigenspace is *at most* 4-dimensional. The generalized 1-eigenspace is 4-dimensional.
   (h) True: if A, B have Jordan form J, change bases from $A \to J \to B$ to see that A, B are similar.
   (i) False: here $\Phi(A, B) = \det(A)\det(B)$, but the determinant is not a linear function on matrices.
   (j) True: this follows from the spectral theorem, since the associated matrix is symmetric.
   (k) False: with $Q = 2I_2$ we see A and $Q^T AQ = 4A$ are congruent, but the eigenvalues of $4A$ are four times the eigenvalues of A.
   (l) True: in the singular value decomposition $A = U\Sigma V^*$, the matrices U and V are unitary (so V^* is also unitary) and Σ is diagonal.

2. Taking determinants we see $\det(B)$ is nonzero and $\det(ABA) = \det(A)^2 \det(B)$ is nonzero, so $\det(A)^2$ hence $\det(A)$ is nonzero, so A is invertible.

3. (a) By properties of associated matrices, since $[T]_\beta^\gamma$ is invertible, the associated transformation is an isomorphism with inverse transformation $S : W \to V$ having matrix $[S]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.
   (b) Take any basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and let $\gamma = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$: since T is an isomorphism, it maps a basis of V to a basis of W, so γ is a basis of W. Then $[T]_\beta^\gamma = I_n$ directly by definition of the associated matrix.

4. (a) Set $\mathbf{w} = T(\mathbf{v})$: then $\langle T(\mathbf{v}), T(\mathbf{v})\rangle = 0$ hence $T(\mathbf{v}) = 0$.
   (b) As proven in class, since V is finite-dimensional, $\ker(T)$ and $\mathrm{im}(T^*)$ are orthogonal complements. If T is one-to-one then $\ker(T) = 0$ so $\mathrm{im}(T^*) = \{\mathbf{0}\}^\perp = V$ hence T^* is onto.
   (c) As proven in class, $\ker(T^*)$ and $\mathrm{im}(T)$ are orthogonal complements. If T is onto then $\mathrm{im}(T) = V$ so $[\mathrm{im}(T)]^\perp = \{\mathbf{0}\}$ hence $\ker(T^*) = \{\mathbf{0}\}$ so T^* is one-to-one.

5. (a) Suppose $T^*T(\mathbf{v}) = \lambda\mathbf{v}$. Then $||T(\mathbf{v})||^2 = \langle T(\mathbf{v}), T(\mathbf{v})\rangle = \langle \mathbf{v}, T^*T(\mathbf{v})\rangle = \langle \mathbf{v}, \lambda\mathbf{v}\rangle = \overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle = \overline{\lambda}\,||\mathbf{v}||^2$. Therefore we see $\overline{\lambda} = ||T(\mathbf{v})||^2 / ||\mathbf{v}||^2 \geq 0$ and thus λ is a nonnegative real number.
   (b) Applying (a) to the linear transformation $T(\mathbf{v}) = A\mathbf{v}$ yields the result immediately.
   (c) The eigenvalues of $I + B^*B$ are 1 plus the eigenvalues of B^*B hence they are all positive. Then $\det(I + B^*B)$ is the product of the eigenvalues (with multiplicity) so it is also positive.

6. (a) Clearly any diagonal matrix D has a square root E (just take square roots of each of its diagonal entries). Then if A is diagonalizable with $Q^{-1}AQ = D$, for $B = QEQ^{-1}$ we have $B^2 = QE^2Q^{-1} = QDQ^{-1} = A$.
   (b) Suppose $B^2 = A$. Then since $A^2 = 0$ we see $B^4 = 0$. If $\mathbf{v}$ is an eigenvector of B with eigenvalue λ then $\mathbf{0} = 0\mathbf{v} = B^4\mathbf{v} = \lambda^4\mathbf{v}$ hence $\lambda = 0$. Thus, the only eigenvalues of B are zero, so the characteristic polynomial of B is $p(t) = t^2$. By Cayley-Hamilton this implies $B^2 = 0$, but $B^2 = A$ which would give $A = 0$, contradiction.

7. Rank is preserved by similarity, so the Jordan form must also have rank 1. If there is a nonzero eigenvalue then it must have a 1×1 block and the rest of the blocks must be 1×1 of eigenvalue 0. Otherwise if all eigenvalues are 0, there must be one 2×2 block and the remaining blocks are 1×1. So the possibilities are either $\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ (for any $\lambda \neq 0$) or $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

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8. Let J be the Jordan form of A with $J = Q^{-1}AQ$. As in class, $e^A = Q^{-1}e^JQ$, so $\det(e^A) = \det(Q^{-1}e^JQ) = \det(e^J)$. From the formula for the exponential of a Jordan block, or because J is upper-triangular, or by the spectral mapping theorem, the diagonal entries of e^J are just the exponentials of the diagonal entries of J; namely, $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}$. Hence $\det(e^J) = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n} = e^{\mathrm{tr}(A)}$, as claimed.

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9. (a) If $T^* = T$ then $\langle T\mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{v}, T^*\mathbf{v}\rangle = \langle \mathbf{v}, T\mathbf{v}\rangle = \overline{\langle T\mathbf{v}, \mathbf{v}\rangle}$ so $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real.

   (b) With $\mathbf{v} = \mathbf{x} + \mathbf{y}$ we have $0 = \langle T\mathbf{v}, \mathbf{v}\rangle = \langle T\mathbf{x} + T\mathbf{y}, \mathbf{x} + \mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{x}\rangle + \langle T\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, \mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{y}\rangle + \langle T\mathbf{y}, \mathbf{x}\rangle$. Likewise with $\mathbf{w} = \mathbf{x} + i\mathbf{y}$ we have $0 = \langle T\mathbf{w}, \mathbf{w}\rangle = \langle T\mathbf{x} + iT\mathbf{y}, \mathbf{x} + i\mathbf{y}\rangle = \langle T\mathbf{x}, \mathbf{x}\rangle - i\langle T\mathbf{x}, \mathbf{y}\rangle + i\langle T\mathbf{y}, \mathbf{x}\rangle + \langle T\mathbf{y}, \mathbf{y}\rangle = -i\langle T\mathbf{x}, \mathbf{y}\rangle + i\langle T\mathbf{y}, \mathbf{x}\rangle$. In order for both of these to be true we must have $\langle T\mathbf{x}, \mathbf{y}\rangle = \langle T\mathbf{y}, \mathbf{x}\rangle = 0$ for all $\mathbf{x}, \mathbf{y}$. But now setting $\mathbf{y} = T\mathbf{x}$ we see $\langle T\mathbf{x}, T\mathbf{x}\rangle = 0$ hence $T\mathbf{x} = \mathbf{0}$, so T is the zero transformation.

   (c) If $\langle T\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$ then similarly to (a) we see $\langle T\mathbf{v}, \mathbf{v}\rangle = \overline{\langle T\mathbf{v}, \mathbf{v}\rangle} = \overline{\langle \mathbf{v}, T^*\mathbf{v}\rangle} = \langle T^*\mathbf{v}, \mathbf{v}\rangle$, so $\langle (T - T^*)\mathbf{v}, \mathbf{v}\rangle = 0$ for all $\mathbf{v}$. Applying part (b) to $T - T^*$ this means $T - T^*$ is the zero transformation, so $T = T^*$ hence T is Hermitian.

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10. (a) Eigenvalues are 1, 11 with eigenspaces spanned by $(-3, 1)$ and $(1, 3)$ respectively.

   (b) Taking $Q = \dfrac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$ with $Q^{-1} = Q^T$ gives $A^n = Q\begin{bmatrix} 1 & 0 \\ 0 & 11^n \end{bmatrix}Q^T = \dfrac{1}{10}\begin{bmatrix} 9 + 11^n & 3(11^n - 1) \\ 3(11^n - 1) & 1 + 9 \cdot 11^n \end{bmatrix}$.

   (c) From the eigenvalue method the solutions are $(y_1, y_2) = C_1(-3, 1)e^x + C_2(1, 3)e^{11x}$.

   (d) $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 10 & 0 & 1 \end{array}\right] \xrightarrow[C_2 - 3/2C_1]{R_2 - 3/2R_1} \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 11/2 & -3/2 & 1 \end{array}\right]$ so we can take $Q = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}$.

   (e) Since the eigenvalues are both positive, the quadratic form is positive definite and the conic is an ellipse.

   (f) Since the eigenvalues are both positive, the critical point is a local minimum.

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11. If T is invertible and all of its singular values equal 1, then the eigenvalues of the Hermitian operator T^*T are the squares of the singular values, which are all 1. Since this operator is diagonalizable, this means the diagonalization is the identity, but this is equivalent to saying T^*T is the identity. Conversely, if T is an isometry, then T^*T is the identity on V so its singular values are the square roots of the eigenvalues of the identity map, so they are all 1.
