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1. Find the following:

   (a) The gcd and lcm of 288 and 600.

   - From the Euclidean algorithm we find $\gcd(288, 600) = \boxed{24}$. Then $\text{lcm}(288, 600) = 288 \cdot 600/24 = \boxed{7200}$.

   (b) The prime factorizations of 2025 and 2026. (You may want to use a calculator for these!)

   - Using trial division or a computer we can find $2025 = \boxed{3^4 5^2}$ and $2026 = \boxed{2 \cdot 1013}$.

   (c) The prime factorizations of 2025^{2026} and 2026^{2025}.

   - Using (b) we see $2025^{2026} = \boxed{3^{8104} 5^{4052}}$ and $2026^{2025} = \boxed{2^{2025} 1013^{2025}}$.

   (d) The prime factorizations of 111, 1001, and 111111.

   - Using trial division we can find $111 = \boxed{3 \cdot 37}$ and $1001 = \boxed{7 \cdot 11 \cdot 13}$.
   - Then $111111 = 111 \cdot 1001 = \boxed{3 \cdot 7 \cdot 11 \cdot 13 \cdot 37}$.

   (e) The gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.

   - From the prime factorizations, the gcd is $\boxed{2^4 3^8 5^7 7^7 11^2}$ (take the minimum of the two exponents) and the lcm is $\boxed{2^8 3^{11} 5^7 7^8 11^{11}}$ (take the maximum of the two exponents).

   (f) The prime factorization of 12! and the number of positive divisors of 12!.

   - Noting $12! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$, the primes appearing are 2, 3, 5, 7, 11.
   - Counting powers of each prime yields $15! = \boxed{2^{10} 3^5 5^2 7 \cdot 11}$.
   - By the number-of-divisors formula, the number of divisors is $(1+10)(1+5)(1+2)(1+1)(1+1) = 11 \cdot 6 \cdot 3 \cdot 2 \cdot 2 = \boxed{792}$.

   (g) The number of divisors and the sum of divisors of 10000.

   - Since $10000 = 2^4 5^4$ we see it has $(4+1)(4+1) = \boxed{25}$ divisors, and the sum of divisors is $(1 + 2 + 2^2 + 2^3 + 2^4)(1 + 5 + 5^2 + 5^3 + 5^4) = \boxed{24211}$.

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2. It is sometimes claimed (occasionally in actual textbooks) that if $p_1, p_2, \ldots, p_k$ are the first k primes, then the number $n = p_1 p_2 \cdots p_k + 1$ used in Euclid's proof is always prime for any $k \geq 1$. Find a counterexample to this statement; make sure to justify that it is actually a counterexample.

   - Although $2 \cdot 3 + 1 = 7$, $2 \cdot 3 \cdot 5 + 1 = 31$, $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$, and $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$ are all prime, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$ is not, since $30031 = 59 \cdot 509$.

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3. Find examples of the following things:

   (a) Four different pairs of positive integers (a, b) with $a \leq b$ such that $\gcd(a, b) = 30$ and $\text{lcm}(a, b) = 1800$.

   - Since $\gcd(a, b) = 30$ we see that $a = 30a'$ and $b = 30b'$ for some relatively prime integers a' and b'.
   - Then $\text{lcm}(a, b) = \text{lcm}(30a', 30b') = 30\text{lcm}(a', b') = 30a'b'$ because a', b' are relatively prime, so this gives $30a'b' = 1800$ hence $a'b' = 60$. So we must simply find pairs of relatively integers (a', b') with $a' \leq b'$ such that $a'b' = 60$. Using the prime factorization $60 = 2^2 \cdot 3 \cdot 5$ we see that each prime power is taken fully by one factor or by the other, so there are four possible choices for (a', b'), namely $(5, 12)$, $(4, 15)$, $(3, 20)$, $(1, 60)$.
   - This yields the pairs $(a, b) = \boxed{(150, 360), (30, 450), (90, 600), (30, 1800)}$.

(b) A positive integer n such that $n/2$ is a perfect square and $n/3$ is a perfect cube.

- If $n = 2^a 3^b$ then we want $2^{a-1}3^b$ to be a square, so both exponents are even, and $2^a 3^{b-1}$ to be a cube, so both exponents are multiples of 3.
- Testing small possibilities shows that $a = 3$, $b = 4$ satisfies all the requirements, so $n = \boxed{2^3 3^4 = 648}$ works. (In fact, it is the smallest example.)

(c) A positive integer n such that n, $n+1$, $n+2$, $n+3$, $n+4$, and $n+5$ all have more than one distinct prime factor.

- We want six consecutive integers that are all composite (and also not prime powers).
- The smallest primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.
- The first gap of six consecutive integers is 90 through 96. Since 90, 91, 92, 93, 94, 95, 96 all have more than one prime factor, we can take $n = \boxed{91}$.
- There are many additional larger options, as well.

(d) An integer that is a multiple of 15 that has exactly 15 positive divisors.

- By the number-of-divisors formula we see that an integer with 15 divisors is either a 14th power of a prime, or is a square of a prime times a fourth power of a prime, since the only possible factorizations are $15 = 15$ or $15 = 3 \cdot 5$.
- But a multiple of 15 has factors of both 3 and 5, so the only options are to take $\boxed{3^2 5^4 = 2025}$ or $\boxed{3^4 5^2 = 5625}$.

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4. Let n be a positive integer greater than 1.

(a) Show that if n is composite, then n must have at least one divisor d with $1 < d \leq \sqrt{n}$. Deduce that if n is composite, then n has at least one *prime* divisor $p \leq \sqrt{n}$. [Hint: Write $n = ab$ where $1 < a \leq b < n$.]

- Since n is composite, we can write $n = ab$ for some $1 < a \leq b < n$.
- Then we see that $a^2 \leq ab = n$, and so $a \leq \sqrt{n}$. Thus, n has a divisor (namely a) that is $\leq \sqrt{n}$ as claimed.
- For the second statement, simply let p be any prime divisor of a: then $1 < p \leq a \leq \sqrt{n}$, and p divides a hence n. Thus, n has a prime divisor $\leq \sqrt{n}$.

(b) Show that if no prime less than or equal to $\sqrt{n}$ divides n, then n is prime.

- This is simply the contrapositive of the second statement in part (a): if n has no prime divisor $p \leq \sqrt{n}$ then n is not composite (i.e., n is prime).

(c) Show explicitly that $n = 109$ and $n = 251$ are prime by verifying that they are not divisible by any prime $\leq \sqrt{n}$.

- Note that $\sqrt{109} < \sqrt{121} = 11$, so for 109 we only have to check the primes 2,3,5,7. But since $109 = 54 \cdot 2 + 1 = 36 \cdot 3 + 1 = 21 \cdot 5 + 4 = 15 \cdot 7 + 4$ we see that none of the primes 2,3,5,7 divide 109, so 109 is prime.
- Likewise, since $\sqrt{251} < \sqrt{256} = 16$, for 251 we only have to check the primes 2,3,5,7,11,13. Since $251 = 2 \cdot 125 + 1 = 3 \cdot 83 + 2 = 50 \cdot 5 + 1 = 35 \cdot 7 + 6 = 22 \cdot 11 + 9 = 19 \cdot 13 + 4$, none of these primes divide 251, so 251 is prime.

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5. Recall that the factorial of n is defined as $n! = n \cdot (n-1) \cdot \dots \cdot 1$, so for example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. (Note that $0!$ is defined to be 1.)

   (a) If $n \geq 3$, show that the integers $n!+2$, $n!+3$, ... , $n!+n$ are all composite.

   - Notice that for any $2 \leq k \leq n$, we have k appearing as a factor in $n!$, and so $n!+k = k \cdot [1 \cdot 2 \cdots (k-1) \cdot (k+1) \cdots n + 1]$.
   - Since the term k and the remaining term $1 \cdot 2 \cdots (k-1) \cdot (k+1) \cdots n + 1$ are both greater than 1, this gives an explicit factorization of $n!+k$, so it is composite as claimed.

   (b) Suppose we list the primes in increasing order: $p_1 < p_2 < p_3 < \cdots$, so, for instance, $p_5 = 11$. For an arbitrary positive integer n, show that there is some pair of consecutive primes p_k, p_{k+1} such that $p_{k+1} - p_k \geq n$: in other words, that there exists a "prime gap" of size at least n.

   - By (a), none of the $n-1$ numbers $n!+2$, $n!+3$, ... , $n!+n$ is prime. So if p_k is the largest prime below $n!+2$ and p_{k+1} is the smallest prime above $n!+n$, then $p_{k+1} - p_k \geq (n!+n+1) - (n!+1) = n$, as required.

   Remark: Part (b) shows that there are arbitrarily large prime gaps. The smallest possible prime gap that could occur infinitely often is 2, and the twin prime conjecture says that there are infinitely many pairs of consecutive primes that differ by 2.

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6. The goal of this problem is to study which numbers of the form $N = a^k - 1$ can be prime, where a and k are positive integers greater than 1.

   (a) Show that if $a > 2$, then $N = a^k - 1$ is not prime.

   - By factoring, we see $N = a^k - 1$ has $N = (a-1)(a^{k-1} + a^{k-2} + \cdots + a + 1)$.
   - Since $a > 1$, both factors are greater than 1, so N is composite.

   (b) Show that if k is composite, then $N = 2^k - 1$ is not prime. [Hint: If $k = rs$, show N is divisible by $2^r - 1$.]

   - By factoring, we see $(2^r)^s - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1)$.
   - Again since $r, s > 1$ we see that both factors are greater than 1, so N is composite.

   (c) Deduce that the only possible primes of the form $N = a^k - 1$ are those of the form $2^p - 1$ where p is a prime. (Such primes are called Mersenne primes.) Are all the numbers of the form $2^p - 1$ (p prime) actually prime?

   - The first part follows from parts (a) and (b), since $a^k - 1$ is composite if $a \neq 2$ and k is nonprime.
   - However, not all numbers of the form $2^p - 1$ with p prime are prime: although $2^3 - 1 = 7$, $2^5 - 1 = 31$, and $2^7 - 1 = 127$ are prime, we have $2^{11} - 1 = 23 \cdot 89$, so $2^p - 1$ is not always prime even when p is prime.

   Remark: Mersenne primes can be used to construct perfect numbers, as described in problem 7.

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7. Recall that if N is a positive integer, then $\sigma(N)$ denotes the sum of the positive divisors of N. We say that N is a perfect number when $\sigma(N) = 2N$: this is often phrased as "the sum of all of the proper divisors of N equals N itself".

   (a) Show that if $2^{n+1} - 1$ is a prime number, then the number $N = 2^n(2^{n+1} - 1)$ is perfect.

   - If $2^{n+1} - 1$ is prime, then by the formula for the sum of divisors, with $N = 2^n(2^{n+1} - 1)$ we have $\sigma(N) = (1 + 2 + \cdots + 2^n)(1 + 2^{n+1} - 1) = (2^{n+1} - 1)2^{n+1} = 2N$, so N is perfect.

   (b) Show that 28, 496, 8128 are perfect numbers.

   - Note that $28 = 2^2 \cdot 7$, $496 = 2^4 \cdot 31$, and $8128 = 2^6 \cdot 127$, so by part (a) since each of $7 = 2^3 - 1$, $31 = 2^5 - 1$, and $127 = 2^7 - 1$ is a prime, we see that 28, 496, and 8128 are perfect numbers.

We would now like to prove a partial converse to (a): namely, that every even perfect number must be of the form given in (a). So suppose $N = 2^n c$ is a perfect number where $n \geq 1$ and c is odd, and observe that $\sigma(N) = (2^{n+1} - 1)\sigma(c)$.

(c) Show that c must be divisible by $2^{n+1} - 1$.

- Since $\sigma(N) = 2N$ the formula for $\sigma(N)$ gives $2^{n+1}c = (2^{n+1} - 1)\sigma(c)$, so $2^{n+1} - 1$ divides $2^{n+1}c$.
- But $2^{n+1} - 1$ is relatively prime to 2^{n+1}, so by the relatively prime divisibility property, it must divide the other factor c.

(d) Show that c must equal $2^{n+1} - 1$ and that $2^{n+1} - 1$ is prime. [Hint: Use $\sigma(c) = 2^{n+1}c/(2^{n+1} - 1)$ and the fact that c has two divisors c and $c/(2^{n+1} - 1)$ to conclude it has no other divisors.]

- By (c) we know that c is divisible by $2^{n+1} - 1$ so c has at least the two divisors c and $c/(2^{n+1} - 1)$.
- But from (c) we also saw that $\sigma(c) = 2^{n+1}c/(2^{n+1} - 1) = c + c/(2^{n+1} - 1)$, and so these are the only two divisors of c. Therefore, the smaller one $c/(2^{n+1} - 1)$ must be 1, hence $c = 2^{n+1} - 1$, and the larger one $c = 2^{n+1} - 1$ must be prime.

(e) Deduce that the even perfect numbers are precisely the numbers of the form $N = 2^n(2^{n+1} - 1)$ where $2^n - 1$ is prime.

- This is immediate from (a) and (d), since (a) showed all these numbers are perfect and (d) shows any even perfect number is of the form $N = 2^n c$ where $c = 2^{n+1} - 1$ is prime.

Remark: Perfect numbers have been of mathematical (and numerological) interest since antiquity. Euclid established the result in (a), and roughly two millennia later, Euler proved the result in (e). It is not known whether there are infinitely many even perfect numbers, and it is also not known whether there are any odd perfect numbers.
