E. Dummit's Math 3527 ~ Number Theory 1, Spring 2026 ~ Homework 10, due Fri Apr 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each Gaussian integer α, find (i) the number of residue classes in $\mathbb{Z}[i]$ modulo α, and (ii) the prime factorization of α in $\mathbb{Z}[i]$:
   (a) $\alpha = 19 + 48i$.
   (b) $\alpha = 28 - 4i$.
   (c) $\alpha = 20 + 7i$.
   (d) $\alpha = 60 - 11i$.
   (e) $\alpha = 2025$.
   (f) $\alpha = 2465$.

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2. Let $R = \mathbb{Z}[i]$ and $r = 4 + 2i$.
   (a) Find the prime factorization of r in $\mathbb{Z}[i]$.
   (b) Determine the total number of residue classes in R/rR.
   (c) Draw a fundamental region for R/rR, and use it to find an explicit list of residue class representatives.

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3. For each integer, determine whether it can be written as a sum of two squares (of integers), and for those that can, give at least one such way:
   (a) The integer 2600.
   (b) The integer 2024.
   (c) The integer 2026.
   (d) The integer 77077.
   (e) The prime $p = 2909$, given $878^2 \equiv -1 \pmod{p}$.
   (f) The prime $p = 5813$, given $796^2 \equiv -1 \pmod{p}$.

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4. Find all of the Pythagorean right triangles (i.e., with integer side lengths) where one side length is 2026.

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5. List all of the (nonzero) quadratic residues, and all of the quadratic nonresidues, modulo 13 and modulo 19.

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6. Calculate the following Legendre symbols (i) using Euler's criterion, and (ii) using quadratic reciprocity.
   (a) $\left(\frac{3}{17}\right)$.
   (b) $\left(\frac{11}{733}\right)$.
   (c) $\left(\frac{-5}{67}\right)$.
   (d) $\left(\frac{67}{101}\right)$.
   (e) $\left(\frac{15}{23}\right)$.
   (f) Which method is easier to implement by hand?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

7. Prove that if an integer is the sum of squares of two rational numbers, then it is the sum of squares of two integers: for example, $5 = (22/13)^2 + (19/13)^2 = 2^2 + 1^2$. [Hint: Clear denominators and use the characterization of sums of two squares.]

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8. We have given a geometric description for finding residue class representatives for $\mathbb{Z}[i]$ modulo α. In certain cases, we can give a more direct description.
   (a) If $\alpha = n$ is an integer (in $\mathbb{Z}$), show that the residue classes modulo α are represented by the elements $c + di$, with $0 \le c \le n-1$ and $0 \le d \le n-1$. [Hint: Draw the fundamental region.]
   (b) If $\pi = a + bi$ is a prime element with $N(\pi) = p$ a prime congruent to 1 modulo 4 (e.g., $\pi = 2 + i$ or $\pi = 3 - 2i$), show that the residue classes modulo π are represented by the elements $0, 1, \dots, p-1$. [Hint: Show $a \in \mathbb{Z}$ is divisible by π only if it is divisible by p; deduce $0, 1, \dots, p-1$ are distinct mod π.]

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9. The goal of this problem is to discuss some results on consecutive integers that are sums of two squares.
   (a) If N is the sum of two squares, show that N must be congruent to 0, 1, 2, 4, or 5 modulo 8.
   (b) Deduce that there do not exist four consecutive integers all of which are the sum of two squares.
   (c) Show that there exist infinitely many N for which N, $N+1$, $N+2$ are all the sum of two squares. [Hint: Try $N = 4a^4 + 4a^2$.]
