E. Dummit's Math 3527 ~ Number Theory 1, Spring 2026 ~ Homework 1, due Fri Jan 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. Use of generative AI in any manner is not allowed on this or any other course assignments.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Match the erroneous proofs (a)-(e) with the reasons (1)-(4) they are not valid inductive proofs of the claims. Some reasons may be used more than once and others not at all.

   Proofs:

   (a) Proposition: If $a_1 = 3$ and $a_{n+1} = 3a_n + 2$ for all $n \geq 1$, then $a_n = 3^n - 1$ for all n.
   Proof: Induct on n. The base case $n = 1$ is trivial. For the inductive step, suppose $a_n = 3^n - 1$. Then $a_{n+1} = 3a_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 1$ as required.

   (b) Proposition: For every positive integer n, $1 + 2 + 4 + \cdots + 2^n = 2n + 1$.
   Proof: Induct on n. The base case $n = 1$ follows because $1 + 2 = 2 \cdot 1 + 1$. For the inductive step, we want to show that $1 + 2 + 4 + \cdots + 2^{n+1} = 2^{n+2} - 1$. Multiplying both sides by 0 yields $0 = 0$, which is a true statement. Therefore the result holds by induction.

   (c) Proposition: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   Proof: Strong induction on n. The base case $n = 1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{n+1} = 4a_n - 4a_{n-1} = 4 \cdot 2^n - 4 \cdot 2^{n-1} = 4 \cdot 2^n - 2 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}$ as required.

   (d) Proposition: All horses are the same color.
   Proof: Induct on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n + 1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.

   (e) Proposition: For every positive integer n, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$.
   Proof: Induct on n. The base case $n = 1$ follows because $1 = \frac{1}{2}(1)(2)$. For the inductive step, suppose that $1+2+3+\cdots+n+(n+1) = \frac{1}{2}(n+1)(n+2)$. Subtracting $n+1$ from both sides yields $1+2+3+\cdots+n = \frac{1}{2}(n+1)(n+2) - (n+1) = \frac{1}{2}n(n+1)$, as required. Therefore the result holds by induction.

   Reasons:

   (1) The proof does not actually check that the base case is correct.

   (2) The proof of the inductive step shows that the claimed result implies a true statement instead of proving the claimed result.

   (3) The proof of the inductive step assumes more base cases than are actually checked.

   (4) The proof of the inductive step shows that the result for $n + 1$ implies the result for n, rather than the other way around.

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2. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ and also to find integers x and y such that $d = ax + by$. (Make sure to include enough detail in your calculations to show you used the Euclidean algorithm.)

   (a) $a = 44$, $b = 12$.

   (b) $a = 481$, $b = 24$.

   (c) $a = 18063$, $b = 2025$.

   (d) $a = 12445$, $b = 5567$.

   (e) $a = 18200$, $b = 3505$.

   (f) $a = 233$, $b = 144$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Prove the following basic properties of divisibility (note that some of these properties are referred to, but not proven, in the course notes; you are expected to give the details of the proof!):

   (a) If a, b are integers, show that $a|b$ if and only if $a|(-b)$.

   (b) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

   (c) If a, b, c are integers such that $a|b$ and $a \nmid c$, show that $a \nmid (b + c)$.

   (d) If a, b, c, x, y are integers such that $a|b$ and $a|c$, show that $a|(xb + yc)$.

   (e) If a, b are integers, show that a, b and $a, a + b$ have the same set of common divisors.

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4. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n. [Hint: Use induction.]

   (b) Prove that $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$ for every positive integer n.

   (c) Prove that $F_{2n+1} = F_{n+1}^2 + F_n^2$ and $F_{2n+2} = F_{n+1}(F_{n+2} + F_n)$ for all $n \geq 1$. [Hint: Show both together by induction.]

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5. The goal of this problem is to study a few more miscellaneous properties of Fibonacci numbers, as defined in problem 4.

   (a) Find $\gcd(F_5, F_{10})$, $\gcd(F_6, F_9)$, $\gcd(F_6, F_{12})$, and $\gcd(F_{12}, F_{13})$.

   (b) Show that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$, for any nonnegative n and k. [Hint: Induct on k.]

   (c) Show that $F_a | F_{na}$ for all positive integers n. [Hint: Use (b).]

   (d) Show that F_a and F_{a+1} are relatively prime for all n.

   Remark: It can in fact be shown using the results in (b), (c), and (d) that the gcd of any two Fibonacci numbers is another Fibonacci number, and more specifically that $\gcd(F_a, F_b) = F_{\gcd(a,b)}$.
