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1. Let $V$ be a finite-dimensional vector space with scalar field $F$ and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $\dim(V) = n$ and $T$ has $n$ distinct eigenvalues in $F$, then $T$ is diagonalizable.

   - $\boxed{\text{True}}$: if $T$ has $n$ distinct eigenvalues, then each eigenspace must have dimension 1. But then each eigenvalue's multiplicity is equal to the dimension of its eigenspace, so $T$ is diagonalizable.

   (b) If $\dim(V) = n$ and $T$ is diagonalizable, then $T$ has $n$ distinct eigenvalues in $F$.

   - $\boxed{\text{False}}$: there are diagonalizable linear transformations with repeated eigenvalues, such as the identity transformation (all its eigenvalues are 1, but it is clearly diagonalizable).

   (c) If $A$ is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

   - $\boxed{\text{True}}$: if $Q^{-1}AQ = D$ is diagonal, then $Q^{-1}(A + I_n)Q = D + I_n$ is also diagonal.

   (d) For any scalar $\lambda$, the $\lambda$-eigenspace of $T$ is a subspace of the generalized $\lambda$-eigenspace of $T$.

   - $\boxed{\text{True}}$: every $\lambda$-eigenvector is a generalized $\lambda$-eigenvector, so the $\lambda$-eigenspace is a subset (hence a subspace) of the generalized $\lambda$-eigenspace.

   (e) For any $\lambda$, a chain of generalized $\lambda$-eigenvectors is linearly independent.

   - $\boxed{\text{True}}$: we proved this in the course of showing that the generalized $\lambda$-eigenspace has a chain basis.

   (f) There always exists a basis $\beta$ of $V$ consisting of generalized eigenvectors of $T$.

   - $\boxed{\text{False}}$: we must also know that the eigenvalues of $T$ all lie in the scalar field $F$. For example, the linear transformation $T(x, y) = (y, -x)$ has no such basis when $F = \mathbb{R}$, since its eigenvalues are $\pm i$.

   (g) If all eigenvalues of $T$ lie in $F$, then there exists a basis $\beta$ of $V$ of generalized eigenvectors for $T$.

   - $\boxed{\text{True}}$: this was proven in class.

   (h) There always exists some basis $\beta$ of $V$ such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.

   - $\boxed{\text{False}}$: we must also know that the eigenvalues of $T$ all lie in the scalar field $F$. For example, the linear transformation $T(x, y) = (y, -x)$ has no such basis when $F = \mathbb{R}$, since its eigenvalues are $\pm i$.

   (i) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.

   - $\boxed{\text{True}}$: here, because the eigenvalues of $A$ all lie in $\mathbb{C}$ (because $\mathbb{C}$ is algebraically closed), we know that $A$ has a Jordan canonical form.

   (j) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.

   - $\boxed{\text{True}}$: if a matrix is diagonalizable then its Jordan canonical form will be the diagonalization.

   (k) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.

   - $\boxed{\text{True}}$: since the matrix is similar to its Jordan form, that means the matrix is similar to a diagonal matrix, which is to say, it is diagonalizable.

   (l) Two matrices are similar if and only if they have equivalent Jordan canonical forms.

   - $\boxed{\text{True}}$: every matrix is similar to its Jordan canonical form, and similarity is transitive.

   (m) If $J$ is the Jordan canonical form of $A$, then $J + I_n$ is the Jordan canonical form of $A + I_n$.

   - $\boxed{\text{True}}$: if $PAP^{-1} = J$ then $P(A + I_n)P^{-1} = J + I_n$, and $J + I_n$ is also in Jordan canonical form.

   (n) If $J$ is the Jordan canonical form of $A$, then $J^2$ is the Jordan canonical form of $A^2$.

   - $\boxed{\text{False}}$: although the Jordan form of $A^2$ is conjugate to $J^2$, $J^2$ need not be in Jordan form.

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2. For each matrix $M \in M_{n\times n}(\mathbb{C})$, find a basis for each of its generalized eigenspaces:

   (a) $\begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}$.

   - The characteristic polynomial is $\det(tI - M) = (t-2)^2$ so the eigenvalues are $\lambda = 2, 2$.
   - We see that $(2I_2 - M)^2 = \begin{bmatrix} -4 & 9 \\ -4 & 8 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so every vector is a generalized 2-eigenvector: thus we can take any basis, such as $\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$.

   (b) $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & 2 \\ 2 & -2 & 0 \end{bmatrix}$.

   - The characteristic polynomial is $\det(tI - M) = (t-1)(t-2)^2$ so the eigenvalues are $\lambda = 1, 2, 2$.
   - First, $I_2 - M = \begin{bmatrix} 0 & -1 & -1 \\ 2 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix} \overset{RREF}{\rightarrow} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 1-eigenbasis $\boxed{\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}}$.
   - $(2I_2 - M)^2 = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -2 & 2 & 2 \end{bmatrix} \overset{RREF}{\rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 2-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$.

   (c) $\begin{bmatrix} 1 & -1 & -5 \\ -1 & 2 & 8 \\ 1 & 0 & -2 \end{bmatrix}$.

   - The characteristic polynomial is $\det(tI - M) = t^2(t-1)$ so the eigenvalues are $\lambda = 0, 0, 1$.
   - First, $(-M)^2 = \begin{bmatrix} -3 & -3 & -3 \\ 5 & 5 & 5 \\ -1 & -1 & -1 \end{bmatrix} \overset{RREF}{\rightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 0-eigenbasis $\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$.
   - Also, $I_2 - M = \begin{bmatrix} 0 & 1 & 5 \\ 1 & -1 & -8 \\ -1 & 0 & 3 \end{bmatrix} \overset{RREF}{\rightarrow} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ giving generalized 1-eigenbasis $\boxed{\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}}$.

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3. Suppose the characteristic polynomial of the $5 \times 5$ matrix $A$ is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of $A$, and list all possible dimensions for each of the corresponding eigenspaces.

   - The eigenvalues are $t = \boxed{0, 0, 0, 1, 1}$. The possible dimensions of the 0-eigenspace are $\boxed{1, 2, 3}$ while the possible dimensions of the 1-eigenspace are $\boxed{1, 2}$.

   (b) Find the determinant and trace of $A$.

   - The determinant is the product of the eigenvalues (with multiplicity), which by (a) is $0^3 1^2 = \boxed{0}$, and the trace is the sum of the eigenvalues (with multiplicity), which by (a) is $3 \cdot 0 + 2 \cdot 1 = \boxed{2}$.

   (c) List all possible Jordan canonical forms of $A$ up to equivalence.

   - For the 0-blocks, the possible sizes are 1-1-1, 2-1, or 3, and for the 1-blocks, the possible sizes are 1-1 or 2.
   - Thus, there are 6 possibilities up to equivalence:

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

(d) If $\ker(A)$ and $\ker(A-I)$ are both 2-dimensional, what is the Jordan canonical form of $A$?

- If $\ker(A)$ is 2-dimensional then the 0-eigenspace has dimension 2, which means there are 2 Jordan blocks with eigenvalue 0, which therefore have sizes 2 and 1.
- By the same logic, there are 2 Jordan blocks with eigenvalue 1, which therefore have sizes 1 and 1.
- So the Jordan canonical form is $J = \boxed{\begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}$.

(e) If $A^3$ is diagonalizable but $A^2$ is not, what is the Jordan canonical form of $A$?

- The Jordan forms of $A^2$ and $A^3$ can be computed from the square and cube of the Jordan form of $A$.
- For the 1-blocks, since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, we see that if $A$ has a 1-block of size 2, then $A^3$ would also have a 1-block of size 2 hence not be diagonalizable. So $A$ must have two 1-blocks of size 1.
- For the 0-blocks, since $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ if $A$ had three blocks of size 1 or blocks of sizes 1 and 2, then $A^2$ would be diagonalizable. However since $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, if the block had size 3 then $A^2$ would not be diagonalizable but $A^3$ would.
- Thus the only possible Jordan canonical form is $J = \boxed{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}}$.

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4. Find the Jordan canonical form of each matrix $A$ over $\mathbb{C}$.

(a) $A = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^2$, so the eigenvalues are $\lambda = 0, 0$.
- We can calculate $\operatorname{rank}(-A) = 1$ and $\operatorname{rank}(-A)^2 = 0$.
- This means there is 1 Jordan block of size 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}}$.
- Alternatively, we could see this just by observing that the matrix is not diagonalizable, since then the only possible Jordan canonical form is the one listed above.

(b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-7)(t^2-7)$, so the eigenvalues are $\lambda = 7, -\sqrt{7}, \sqrt{7}$.
- Since the matrix is diagonalizable (either from the eigenvalue list, or because it is a real symmetric matrix), the Jordan form is the diagonalization $\boxed{\begin{bmatrix} 7 & & \\ & \sqrt{7} & \\ & & -\sqrt{7} \end{bmatrix}}$.

(c) $A = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^2 - 12t + 37$, so the eigenvalues are $\lambda = 6 \pm i$.
- Since the eigenvalues are distinct, the matrix is diagonalizable, and the diagonalization $\boxed{\begin{bmatrix} 6+i & \\ & 6-i \end{bmatrix}}$ is also the Jordan canonical form.

(d) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)(t-2)^2$, so the eigenvalues are $\lambda = 1, 2, 2$.
- Since 1 is a single eigenvalue, it must appear in a block of size 1.
- Also, we can calculate $\text{rank}(2I - A) = 2$ and $\text{rank}(2I - A)^2 = \text{rank}(2I - A)^3 = 1$.
- This means there is 1 Jordan 1-block of size 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 1 & & \\ & 2 & 1 \\ & & 2 \end{bmatrix}}$.
- Alternatively, we could see this just by observing that the matrix is not diagonalizable, since then the only possible Jordan canonical form is the one listed above.

(e) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^3 - 6t^2 + 12t - 8$, whose roots are $\lambda = 2, 2, 2$.
- This means all Jordan blocks have eigenvalue 2. To find the sizes, we calculate $\text{rank}(2I - A) = 2$, $\text{rank}(2I - A)^2 = 1$, $\text{rank}(2I - A)^3 = 0$. So there is only one Jordan block and it has size 3, so the Jordan canonical form is $\boxed{\begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}}$.

(f) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = t^3 - 6t^2 + 12t - 8$, whose roots are $\lambda = 2, 2, 2$.
- This means all Jordan blocks have eigenvalue 2. To find the sizes, we calculate $\text{rank}(2I - A) = 1$, $\text{rank}(2I - A)^2 = 0$.
- This means there are two Jordan blocks of sizes 1 and 2, so the Jordan canonical form is $\boxed{\begin{bmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{bmatrix}}$.

(g) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)^3(t-2)$ so the eigenvalues are $\lambda = 1, 1, 1, 2$.
- Since 2 is a single eigenvalue, it must occur in a single Jordan block of size 1.
- Also, we compute $\text{rank}(A - I) = 2$, $\text{rank}(A - I)^2 = \text{rank}(A - I)^3 = 1$. This means that for $\lambda = 1$, there is Jordan block of size 1 and one block of size 2.
- Hence the Jordan form is $\boxed{\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}}$.

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5. The goal of this problem is to find the Jordan form of the $n \times n$ "all 1s" matrix over an arbitrary field $F$. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

   (a) Show that the 0-eigenspace of $A$ has dimension $n-1$ and find a basis for it.
   - Note that the 0-eigenspace is the same as the nullspace, which we can find by row-reducing.
   - The reduced row-echelon form of $A$ is clearly $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, which has $n-1$ non-pivot columns.
   - Thus, the 0-eigenspace has dimension $n-1$ and has a basis $(1, -1, 0, \dots, 0)$, $(1, 0, -1, 0, \dots, 0)$, ... , $(1, 0, 0, \dots, 0, -1)$.

   (b) If the characteristic of $F$ does not divide $n$, find the remaining nonzero eigenvalue of $A$ and a basis for the corresponding eigenspace, and show that $A$ is diagonalizable. [Hint: Calculate the trace of $A$.]
   - Since the characteristic polynomial has degree $n$, there must be exactly one additional eigenvalue.
   - Since the trace of $A$ is equal to $n$, we see that the sum of all the eigenvalues is $n$, so the other eigenvalue must be $n$.
   - It is then quite easy to see that $(1, 1, \dots, 1)$ is an $n$-eigenvector for $A$.
   - There are thus two eigenspaces, the 0-eigenspace of dimension $n-1$ and the $n$-eigenspace of dimension 1. Since the sum of the eigenspace dimensions is $n$, $A$ is diagonalizable.

   (c) If the characteristic of $F$ does divide $n$, show that $A$ is not diagonalizable, and find its Jordan canonical form. [Hint: Note that char$(F)$ dividing $n$ is the same as saying that $n = 0$ in $F$.]
   - By the same logic as in part (b), since the trace of $A$ is equal to $n$, the other eigenvalue must be $n$.
   - But now $n = 0$ in $F$, so in fact there is only one eigenvalue, namely, $\lambda = 0$.
   - From the calculation in (a), the 0-eigenspace only has dimension $n-1$, so since it is now the only eigenspace, we conclude that $A$ is not diagonalizable.
   - For the Jordan form, we see that all eigenvalues are 0 and that the dimension of the 0-eigenspace is $n-1$. The only possibility, therefore, is that there are $n-2$ Jordan blocks of size 1 and 1 Jordan block of size 2, meaning the Jordan form is $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$.

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6. Suppose $V$ is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.

   (a) Show that the only possible eigenvalues of $T$ are 0 and 1.
   - Suppose $\mathbf{v}$ is an eigenvector of $T$ with eigenvalue $\lambda$, so that $T(\mathbf{v}) = \lambda \mathbf{v}$.
   - Then $\lambda^2 \mathbf{v} = T^2(\mathbf{v}) = T(\mathbf{v}) = \lambda \mathbf{v}$, so $(\lambda^2 - \lambda)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$ this means $\lambda^2 = \lambda$, so $\lambda = 0, 1$.

   (b) Show that $T$ is diagonalizable. [Hint: See homework 6.]
   - As shown in problem 5 of homework 6, if $\beta$ is a basis for $\ker(T)$ followed by a basis for $\mathrm{im}(T)$, then $[T]_\beta^\beta$ is diagonal with diagonal entries all 0s (for the kernel elements) and 1s (for the image elements). This $\beta$ is a diagonalizing basis for $T$.

   (c) Suppose $A$ and $B$ are projection maps on $V$ of the same rank. Show that $A$ and $B$ are similar. Deduce that up to equivalence given by similarity, there are $\dim(V) + 1$ different projection maps on $V$.
   - From (a) and (b) together, we can characterize a projection map up to similarity by its diagonalization, whose diagonal entries must be either 0s or 1s.
   - If $\dim(V) = n$, there are clearly $n+1$ such matrices (with $n$ zero entries, $n-1$ zero entries, ... , 1 zero entry, 0 zero entries) and each of these matrices has a different rank: namely, 0, 1, ... , $n-1$, $n$. Thus, up to similarity, there are $n+1$ different projection maps on $V$.

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7. Let $A \in M_{n\times n}(\mathbb{C})$.

(a) Show that any Jordan-block matrix is similar to its transpose. [Hint: Reverse the Jordan basis.]

- Suppose $J$ is the associated matrix $[T]_\beta^\beta$ for a linear transformation $T$ with ordered basis $\beta = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$: then $T\mathbf{v}_0 = \lambda\mathbf{v}_0$ and $T\mathbf{v}_k = \lambda\mathbf{v}_k + \mathbf{v}_{k-1}$.
- Therefore, with the ordered basis $\gamma = \{\mathbf{v}_n, \mathbf{v}_{n-1}, \ldots, \mathbf{v}_0\}$, we see $[T]_\gamma^\gamma = \begin{bmatrix} \lambda & & & \\ 1 & \cdots & & \\ & \cdots & \lambda & \\ & & 1 & \lambda \end{bmatrix}$. Hence $J$ is similar to its transpose. (Explicitly, $J^T = Q^{-1}JQ$, where $Q$ is the "backwards diagonal" matrix.)

(b) If $J$ is a matrix in Jordan canonical form, show that $J$ is similar to its transpose.

- Suppose $J$ is in Jordan canonical form with blocks $J_1, \ldots, J_d$.
- By part (a) each of the Jordan blocks is similar to its transpose: say, with $J_i^T = Q_i^{-1}J_iQ_i$.
- Then for $Q = \begin{bmatrix} Q_1 & & \\ & \cdots & \\ & & Q_d \end{bmatrix}$ we see $Q^{-1}JQ = \begin{bmatrix} Q_1^{-1} & & \\ & \cdots & \\ & & Q_d^{-1} \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \cdots & \\ & & J_d \end{bmatrix} \begin{bmatrix} Q_1 & & \\ & \cdots & \\ & & Q_d \end{bmatrix} =$ $\begin{bmatrix} Q_1^{-1}J_1Q_1 & & \\ & \cdots & \\ & & Q_d^{-1}J_dQ_d \end{bmatrix} = \begin{bmatrix} J_1^T & & \\ & \cdots & \\ & & J_d^T \end{bmatrix} = J^T$. Thus, $J$ is similar to its transpose.

(c) Show that $A$ is similar to its transpose.

- By definition, $A$ is similar to its Jordan canonical form $J$.
- By part (b), $J$ is similar to $J^T$, and then by taking transposes, $J^T$ is similar to $A^T$, since if $A = Q^{-1}JQ$ then $A^T = (Q^{-1}JQ)^T = Q^TJ^T(Q^{-1})^T = Q^TJ^T(Q^T)^{-1}$. Thus, $A$ is similar to $A^T$.

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8. [Challenge] The goal of this problem is to prove various results about eigenvalues of complex matrices and stochastic matrices. Let $A \in M_{n\times n}(\mathbb{C})$, define $\rho_i(A)$ to be the sum of the absolute values of the entries in the $i$th row of $A$, and define $\rho(A) = \max_{1\le i\le n} \rho_i(A)$.

(a) Define the $i$th Gershgorin disk $C_i$ to be the disc in $\mathbb{C}$ centered at $a_{i,i}$ with radius $r_i(A) = \rho_i(A) - |a_{i,i}|$. Prove Gershgorin's disc theorem: every eigenvalue of $A$ is contained in one of the Gershgorin disks of $A$. [Hint: If $\mathbf{v} = (x_1, \ldots, x_n)$ is an eigenvector where $x_k$ has the largest absolute value among the entries of $\mathbf{v}$, show that $|\lambda x_k - a_{k,k}x_k| \le r_i(A)\,|x_k|$ by noting that $\lambda x_k$ is the $k$th component of $A\mathbf{v}$.]

- Suppose $\mathbf{v} = (x_1, \ldots, x_n)$ is an eigenvector with eigenvalue $\lambda$: then by taking the $k$th component of $A\mathbf{v} = \lambda\mathbf{v}$ we see that $\sum_{j=1}^n a_{k,j}x_j = \lambda x_k$.
- Thus, $|\lambda x_k - a_{k,k}x_k| = \left|\sum_{j=1}^n a_{k,j}x_j - a_{k,k}x_j\right| = \left|\sum_{j\ne k} a_{k,j}x_j\right| \le \sum_{j\ne k} |a_{k,j}|\,|x_j| \le \sum_{j\ne k} |a_{k,j}|\,|x_k| = (\rho_i(A) - |a_{k,k}|)\,|x_k| = r_i(A)\,|x_k|$, where we used the triangle inequality at the first $\le$ and the fact that $|x_j| \le |x_k|$ for each $j$ in the second $\le$.
- Since $|x_k| > 0$ because $\mathbf{v} \ne \mathbf{0}$, dividing through by $|x_k|$ yields $|\lambda - a_{k,k}| \le r_i(A)$: in other words, $\lambda$ lies within a Gershgorin disk of $A$. This holds for all eigenvalues $\lambda$, so all eigenvalues lie within Gershgorin disks of $A$.

(b) For any eigenvalue $\lambda$ of $A \in M_{n\times n}(\mathbb{C})$, prove that $|\lambda| \le \rho(A)$.

- By Gershgorin's disc theorem from (a), we have $|\lambda - a_{k,k}| \le \rho_k(A) - |a_{k,k}|$ for some $k$.
- Then by the triangle inequality, we have $|\lambda| \le |\lambda - a_{k,k}| + |a_{k,k}| = \rho_k(A)$. Since the maximum of a set is greater than or equal to all of the elements, this immediately yields $|\lambda| \le \max_{1\le i\le n} \rho_i(A)$.

(c) Prove that if $A \in M_{n\times n}(\mathbb{R})$ has positive entries and there exists an eigenvalue $\lambda$ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ and the $\lambda$-eigenspace is 1-dimensional and spanned by the vector $\mathbf{v} = (1, 1, \ldots, 1)$. [Hint: Analyze when equality can hold in (a) and (b).]

- If $|\lambda| = \rho(A)$ in part (b), then we must have equality in the triangle inequality: $|\lambda| = |\lambda - a_{k,k}| + |a_{k,k}|$. This occurs if and only if $\lambda$ is real and $\lambda \ge a_{k,k}$, so since $a_{k,k}$ this means $\lambda$ is a positive real number. Since $\rho(A)$ is also a positive real number, this means $\lambda = \rho(A)$.

- Furthermore, if $\mathbf{v} = (x_1, x_2, \ldots, x_k)$ is a corresponding eigenvector (which is necessarily real, since $\lambda$ is real), then to get equality in the argument for (a), we must have $\left|\sum_{j \neq k} a_{k,j} x_j\right| = \sum_{j \neq k} |a_{k,j}| \, |x_j|$ and also $|x_j| = |x_k|$ for each $j$. The first equality requires equality in the triangle inequality, meaning that all of the terms $a_{k,j} x_j$ have the same sign, and the second equality requires all of the $x_j$ to have the same absolute value.
- Since all of the entries of $A$ are positive, these two statements together are equivalent to $x_1 = x_2 = \cdots = x_n$, meaning that $\mathbf{v}$ is a scalar multiple of $(1, 1, \ldots, 1)$. This means that the $\lambda$-eigenspace is 1-dimensional and spanned by $\mathbf{v} = (1, 1, \ldots, 1)$.

(d) If $M$ is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1), show that every eigenvalue $\lambda$ of $M$ has $|\lambda| \leq 1$. Also show that if $M$ has all entries positive, then the only eigenvalue of $M$ of absolute value 1 is $\lambda = 1$, and the 1-eigenspace has dimension 1. [Hint: Consider $M^T$.]

- Note that $M^T$ has rows with nonnegative entries all summing to 1, and it has the same eigenvalues as $M$. Thus, $\rho_i(M^T) = 1$ for each $i$, so by (b), we immediately obtain $|\lambda| \leq 1$.
- Furthermore, since $M^T \mathbf{v} = \mathbf{v}$ where $\mathbf{v} = (1, 1, \ldots, 1)$ we see that 1 is indeed an eigenvalue of $M^T$ hence also of $M$.
- Therefore, by (c), all other eigenvalues of $M$ have absolute value less than 1, and the 1-eigenspace of $M^T$ (hence also of $M$) is 1-dimensional.

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