1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $T(\mathbf{v}) = \lambda\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.

   - $\boxed{\text{False}}$: we would need to exclude $\mathbf{v} = \mathbf{0}$ here, since by definition $\mathbf{v} = \mathbf{0}$ is not an eigenvector.

   (b) Every linear transformation on V has at least one eigenvector.

   - $\boxed{\text{False}}$: there are linear transformations with no eigenvectors, e.g., integration on $\mathbb{R}[x]$.

   (c) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.

   - $\boxed{\text{False}}$: the characteristic polynomial may have no roots in the scalar field. For example, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (y, -x)$ has no real eigenvalues hence no eigenvectors.

   (d) Any two eigenvectors of T are linearly independent.

   - $\boxed{\text{False}}$: this is only true if the associated eigenvalues are different.

   (e) The sum of two eigenvectors of T is also an eigenvector of T.

   - $\boxed{\text{False}}$: usually not, e.g, for $T(x, y) = (x, 2y)$ then $(1, 0)$ and $(0, 1)$ are eigenvectors, but $(1, 1)$ is not.

   (f) The sum of two eigenvalues of T is also an eigenvalue of T.

   - $\boxed{\text{False}}$: usually not, e.g., for $T(x, y) = (x, 2y)$ then 1 and 2 are eigenvalues, but 3 is not.

   (g) If two matrices are similar, then they have the same eigenvectors.

   - $\boxed{\text{False}}$: as we showed, if $A = QBQ^{-1}$ and $A\mathbf{v} = \lambda\mathbf{v}$ then $B(Q\mathbf{v}) = \lambda(Q\mathbf{v})$. This means the eigenspaces of A and B are related by left-multiplication by Q, but are not necessarily equal.

   (h) If two matrices have the same eigenvalues, then they are similar.

   - $\boxed{\text{False}}$: for example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same eigenvalues but are not similar (the latter is not diagonalizable but the former is).

   (i) If two matrices are similar, then they have the same eigenvalues.

   - $\boxed{\text{True}}$: similar matrices have the same characteristic polynomial hence the same eigenvalues.

   (j) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.

   - $\boxed{\text{True}}$: the characteristic polynomial $p(t) = \det(tI - A)$ has degree n hence at most n roots in F.

   (k) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.

   - $\boxed{\text{False}}$: the characteristic polynomial may have repeated roots, in which case it would have fewer than n distinct roots. It may also have irreducible terms of degree > 1, which would further lower the number of roots.

   (l) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.

   - $\boxed{\text{False}}$: although 1 is a double root of the characteristic polynomial, this means only that the 1-eigenspace can have dimension 1 or 2.

   (m) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.

   - $\boxed{\text{True}}$: such a vector would be an element of the 3-eigenspace, but since 3 is not an eigenvalue of A, the 3-eigenspace is trivial.

   (n) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

   - $\boxed{\text{True}}$: $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if $[T]_\beta^\beta$ is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$.

2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable over F and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

(a) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-3 & -1 \\ 2 & t-5 \end{vmatrix} = t^2 - 8t + 17$.
- The roots of this polynomial are $\lambda = 4 \pm i$. Thus, there are $\boxed{\text{no eigenvalues over } \mathbb{R}}$, and so it is $\boxed{\text{not diagonalizable}}$.

(b) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial was calculated above as $p(t) = t^2 - 8t + 17$.
- Over $\mathbb{C}$, the eigenvalues are $\lambda = \boxed{4+i,\ 4-i}$ with respective eigenbases $\boxed{\begin{bmatrix} 1-i \\ 2 \end{bmatrix}}$ and $\boxed{\begin{bmatrix} 1+i \\ 2 \end{bmatrix}}$.
- The matrix $\boxed{\text{is diagonalizable}}$: we can take $D = \begin{bmatrix} 4+i & 0 \\ 0 & 4-i \end{bmatrix}$ and $Q = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix}$.

(c) The matrix $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t-1 & -1 & 1 \\ 2 & t-3 & 2 \\ 1 & 0 & t-1 \end{vmatrix} = (t-1)(t-2)^2$.
- Thus, the eigenvalues are $\lambda = \boxed{1, 2, 2}$.
- Row-reducing $\lambda I - A$ yields the 1-eigenspace basis $\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$, and the 2-eigenspace basis $\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$.
- Since the 2-eigenspace is only 1-dimensional, the matrix is $\boxed{\text{not diagonalizable}}$.

(d) The matrix $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial is $\det(tI - A) = (t-1)(t-2)^2$. Thus, the eigenvalues are $\lambda = \boxed{1, 2, 2}$.
- Row-reducing $\lambda I - A$ yields the 1-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$ and the 2-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}$.
- Since the sum of the eigenspace dimensions is 3, the matrix $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ via the matrix $Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

(e) The matrix $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.

- The characteristic polynomial is $\det(tI - A) = \begin{vmatrix} t+5 & -9 \\ 4 & t-7 \end{vmatrix} = t^2 - 2t + 1$ with roots $\lambda = \boxed{1, 1}$.
- Row-reducing $\lambda I - A = I - A = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$ yields $\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$, with nullspace basis $\boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$.
- Since the 1-eigenspace is only 1-dimensional, the matrix is $\boxed{\text{not diagonalizable}}$.

(f) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

- The characteristic polynomial is $\det(tI - A) = (t-6)(t^2-3)$ with roots $\lambda = \boxed{6, \sqrt{3}, -\sqrt{3}}$.
- Row-reducing $\lambda I - A$ for each of the three possible eigenvalues λ yields that each eigenspace is 1-dimensional.
- Explicitly: 6-eigenbasis $\boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$, $\sqrt{3}$-eigenbasis $\boxed{\begin{bmatrix} -1-\sqrt{3} \\ -1+\sqrt{3} \\ 2 \end{bmatrix}}$, $-\sqrt{3}$-eigenbasis $\boxed{\begin{bmatrix} -1+\sqrt{3} \\ -1-\sqrt{3} \\ 2 \end{bmatrix}}$.
- Since the sum of the eigenspace dimensions is 3, the matrix $\boxed{\text{is diagonalizable}}$.
- The diagonalization is $\begin{bmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{bmatrix}$ via the matrix $Q = \begin{bmatrix} 1 & -1-\sqrt{3} & -1+\sqrt{3} \\ 1 & -1+\sqrt{3} & -1-\sqrt{3} \\ 1 & 2 & 2 \end{bmatrix}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

(a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.

- With respect to the standard basis β the associated matrix is $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = \det(tI - A) = t^2 - 6t - 1 = (t-7)(t+1)$. Hence the eigenvalues are $\lambda = \boxed{7, -1}$, and row-reducing yields corresponding eigenbases $\boxed{(2,3)}$ and $\boxed{(-2,1)}$.
- Since the sum of the eigenspace dimensions is 2, the transformation $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$ via the basis $\beta = \boxed{\{(2,3), (-2,1)\}}$.

(b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.

- With respect to the standard basis $\beta = \{1, x, x^2\}$, the associated matrix is $[D]_\beta^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
- Since this matrix is upper-triangular, the eigenvalues are just the diagonal elements $\lambda = \boxed{0, 0, 0}$. Row-reducing yields that the 0-eigenspace is 1-dimensional and has basis $\boxed{\{1\}}$.
- Since the eigenspace is only 1-dimensional, the transformation is $\boxed{\text{not diagonalizable}}$.

(c) The transpose map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(M) = M^T$.

- With standard basis $\beta = \{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$, the associated matrix is $A = [T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = \det(tI - A) = (t-1)^3(t+1)$. Hence the eigenvalues are $\lambda = \boxed{1, 1, 1, -1}$.
- Row-reducing yields corresponding eigenbases $\boxed{\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}}$ and $\boxed{\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}}$. Alternatively, one could observe that the 1-eigenspace is the space of symmetric matrices, while the (-1)-eigenspace is the space of skew-symmetric matrices.
- Since the sum of the eigenspace dimensions is 4, the transformation $\boxed{\text{is diagonalizable}}$: the diagonalization is $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$ via the basis $\beta = \boxed{\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}}$.

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4. Let $V = C[0, 2\pi]$ with inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$. Also define $\varphi_0(x) = \dfrac{1}{\sqrt{2\pi}}$, and for positive integers k set $\varphi_{2k-1}(x) = \dfrac{1}{\sqrt{\pi}}\cos(kx)$ and $\varphi_{2k}(x) = \dfrac{1}{\sqrt{\pi}}\sin(kx)$.

(a) Show that $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ is an orthonormal set in V. [Hint: Use the product-to-sum identities.]

- First observe that $\int_0^{2\pi}\sin(nx)\,dx = 0 = \int_0^{2\pi}\cos(nx)\,dx$ for any integer $n \neq 0$. Thus, $\langle \varphi_0, \varphi_k\rangle = 0$ for any $k > 0$.
- Furthermore, using the product-to-sum identities, we can write

$$\begin{aligned}
\varphi_{2a}\varphi_{2b} &= \frac{1}{\pi}\sin(ax)\sin(bx) = \frac{1}{2\pi}\left[\cos(a-b)x - \cos(a+b)x\right] \\
\varphi_{2a-1}\varphi_{2b} &= \frac{1}{\pi}\cos(ax)\sin(bx) = \frac{1}{2\pi}\left[\sin(a-b)x - \sin(a+b)x\right] \\
\varphi_{2a-1}\varphi_{2b-1} &= \frac{1}{\pi}\cos(ax)\cos(bx) = \frac{1}{2\pi}\left[\cos(a-b)x + \cos(a+b)x\right]
\end{aligned}$$

  and so when $a \neq b$, each inner product $\langle \varphi_{2a}, \varphi_{2b}\rangle$, $\langle \varphi_{2a-1}, \varphi_{2b}\rangle$, and $\langle \varphi_{2a-1}, \varphi_{2b-1}\rangle$ is zero because both terms integrate to zero (the second also integrates to zero when $a = b$).
- Furthermore, we have $\langle \varphi_0, \varphi_0\rangle = \dfrac{1}{2\pi}\int_0^{2\pi} 1dx = 1$, $\langle \varphi_{2k-1}, \varphi_{2k-1}\rangle = \dfrac{1}{\pi}\int_0^{2\pi}\cos^2(kx)\,dx = 1$, and $\langle \varphi_{2k}, \varphi_{2k}\rangle = \dfrac{1}{\pi}\int_0^{2\pi}\sin^2(kx)\,dx = 1$. Thus, the set is orthonormal.

(b) Let $f(x) = x$. Find $||f||$ and $\langle f, \varphi_n\rangle$ for each $n \geq 0$. (You don't need to give details of the integral evaluations, just the resulting values.)

- We compute $||f|| = \sqrt{\int_0^{2\pi} x^2\,dx} = \sqrt{\dfrac{8\pi^3}{3}}$, $\langle \varphi_0, f\rangle = \int_0^{2\pi}\dfrac{x}{\sqrt{2\pi}}\,dx = \sqrt{2\pi^3}$, with $\langle \varphi_{2k-1}, f\rangle = \int_0^{2\pi}\dfrac{x}{\sqrt{\pi}}\cos(kx)\,dx = 0$, and $\langle \varphi_{2k}, f\rangle = \int_0^{2\pi}\dfrac{x}{\sqrt{\pi}}\sin(kx)\,dx = -\dfrac{2\sqrt{\pi}}{k}$ for $k \geq 1$.

(c) With $f(x) = x$, assuming that $f(x) = \sum_{k=0}^{\infty}\langle f, \varphi_k\rangle\,\varphi_k(x)$, derive Leibniz's formula $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$. [Hint: Set $x = \pi/2$.]

- By part (b), the formula yields $f(\pi/2) = \pi - \sum_{k=1}^{\infty}\dfrac{2}{k}\sin(k\pi/2) = \pi - \left[2 - \dfrac{2}{3} + \dfrac{2}{5} - \dfrac{2}{7} + \cdots\right]$. Since $f(\pi/2) = \pi/2$, rearranging yields $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots = \dfrac{\pi}{4}$ as claimed.

(d) With $f(x) = x$, assuming that $||f||^2 = \sum_{k=0}^{\infty}\langle f, \varphi_k\rangle^2$ (see problem 10 of the midterm for why this is a reasonable statement), find the exact value of $\sum_{k=1}^{\infty}\dfrac{1}{k^2}$.

- By part (b), the formula yields $2\pi^3 + \sum_{k=1}^{\infty}\left(-\dfrac{2\sqrt{\pi}}{k}\right)^2 = \dfrac{8\pi^3}{3}$, so that $\sum_{k=1}^{\infty}\dfrac{4\pi}{k^2} = \dfrac{2\pi^3}{3}$.
- Dividing by 4π yields $\sum_{k=1}^{\infty}\dfrac{1}{k^2} = \boxed{\dfrac{\pi^2}{6}}$. (This is, in fact, the actual value of this sum!)

Remarks: The identity $||f||^2 = \sum_{k=0}^{\infty}\langle f, \varphi_k\rangle^2$ is known as <u>Parseval's identity</u>. The problem of computing the value of the infinite sum $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is known as the Basel problem. The correct value was (famously) first identified by Euler, who evaluated the sum by decomposing the function $\frac{\sin \pi x}{\pi x}$ as the infinite product $\prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2})$ and then comparing the power series coefficients of both sides.

5. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

(a) Find all of the eigenvalues and a basis for each eigenspace of L.

- Solving $L(a_1, a_2, a_3, a_4, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ gives $(a_2, a_3, a_4, a_5, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ whence $a_2 = \lambda a_1,\ a_3 = \lambda a_2,\ \dots\ ,\ a_{i+1} = \lambda a_i,\ \dots$.
- It is then easy to see that the λ-eigenspace is 1-dimensional and spanned by the vector $(1, \lambda, \lambda^2, \lambda^3, \dots)$. In particular, every element $\lambda \in F$ is an eigenvalue of L.

(b) Find all of the eigenvalues and a basis for each eigenspace of R.

- Solving $R(a_1, a_2, a_3, a_4, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ gives $(0, a_1, a_2, a_3, \dots) = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \dots)$ whence $\lambda a_1 = 0,\ \lambda a_2 = a_1,\ \dots\ ,\ \lambda a_{i+1} = a_i$.
- If $\lambda \neq 0$ then cancelling λ gives $a_i = 0$ for all i (but the zero vector is not an eigenvector by definition), while if $\lambda = 0$ then again we see $a_i = 0$ for all i.
- Thus R has no eigenvalues since $(a_1, a_2, a_3, \dots) = \lambda(0, a_1, a_2, a_3, \dots)$ forces $a_1 = a_2 = a_3 = \cdots = 0$.

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6. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.

(a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.

- Suppose $T\mathbf{v} = \lambda\mathbf{v}$. Then $\lambda(S\mathbf{v}) = S(\lambda\mathbf{v}) = S(T\mathbf{v}) = T(S\mathbf{v})$ so $S\mathbf{v}$ is also a λ-eigenvector of T.

(b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

- Suppose $T\mathbf{v} = \lambda\mathbf{v}$. Then $T^2\mathbf{v} = T(T\mathbf{v}) = T(\lambda\mathbf{v}) = \lambda(T\mathbf{v}) = \lambda^2\mathbf{v}$.
- By repeating this argument (equivalently, by a trivial induction) we see that $T^n\mathbf{v} = \lambda^n\mathbf{v}$, so $\mathbf{v}$ is also an eigenvector of T^n with corresponding eigenvalue λ^n.

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7. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

(a) If $B = -\dfrac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

- By the Cayley-Hamilton theorem, we know that $A^n + a_{n-1}A^{n-1} + \cdots + a_2 A^2 + a_1 A + a_0 I_n = 0$, so rearranging yields $A^n + a_{n-1}A^{n-1} + \cdots + a_2 A^2 + a_1 A = -a_0 I_n$.
- From the definition of B, we can multiply through by A to see that $AB = -\dfrac{1}{a_0}(A^n + a_{n-1}A^{n-1} + \cdots + a_2 A^2 + a_1 A) = -\dfrac{1}{a_0}(-a_0 I_n) = I_n$, as claimed.

(b) Show that there exists a polynomial $q(x)$ of degree at most $n - 1$ such that $A^{-1} = q(A)$.

- By part (a) we see that $AB = I_n$ so that $B = A^{-1}$.
- The desired statement is then immediate from the expression in part (a), since the expression for B is a polynomial in A of degree $n - 1$.

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8. [Challenge] The goal of this problem is to give some counterexamples for results about orthogonal complements, projections, best approximations, and adjoints in infinite-dimensional spaces. Let V be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ with only finitely many nonzero terms, with inner product given by $\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i b_i$. (Note that this sum converges since only finitely many terms are nonzero.) Let $\mathbf{e}_i$ be the ith unit coordinate vector and observe that $\{\mathbf{e}_i\}_{i\geq 1}$ is an orthonormal basis for V. Now for each $n \geq 2$, let $\mathbf{v}_n = \mathbf{e}_1 - \mathbf{e}_n$ and define $W = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \dots)$.

   (a) Show that $\mathbf{e}_1 \notin W$ so that W is a proper subspace of V, but that $W^\perp = \{\mathbf{0}\}$.

   - If we have $\mathbf{e}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$, then expanding out yields $\mathbf{e}_1 = (a_1 + \cdots + a_n)\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3 - \cdots - a_n\mathbf{e}_n$ so since the $\mathbf{e}_i$ are linearly independent, we would have $a_1 + \cdots + a_n = 1$ and also $a_2 = a_3 = \cdots = a_n = 0$, but this is contradictory. Thus $\mathbf{e}_1 \notin W$.
   - Next, if $\mathbf{w} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n$ is an element of $W^\perp$, we have $0 = \langle b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n, \mathbf{v}_k \rangle = b_1 - b_k$. Thus $b_k = b_1$ for all k, but since only finitely many b_k can be nonzero, we must have $b_1 = b_2 = \cdots = 0$, and so $\mathbf{w} = \mathbf{0}$.

   (b) Show that $W^\perp + W \neq V$ and that $(W^\perp)^\perp \neq W$.

   - By (a) we have $W^\perp + W = W \neq V$, and also $(W^\perp)^\perp = (\{\mathbf{0}\})^\perp = V \neq W$.

   (c) For any $\mathbf{v} \notin W$, show that there does not exist any choice of $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. Conclude that there is not a well-defined orthogonal projection map of V onto W.

   - Suppose we had such $\mathbf{w}, \mathbf{w}^\perp$. From (a) we know that $W^\perp = \{\mathbf{0}\}$, so the only possible choice would be $\mathbf{w}^\perp = \mathbf{0}$. But this would imply $\mathbf{v} = \mathbf{w}$ which is impossible since $\mathbf{v}$ is not in W.

   (d) Show that there exists a vector $\mathbf{w}_n \in W$ such that $||\mathbf{w}_n - \mathbf{e}_1|| = 1/n$ for any positive integer n. Deduce that there is no possible best approximation vector $\mathbf{w}$ to $\mathbf{e}_1$ inside W (namely with $||\mathbf{w} - \mathbf{e}_1|| \leq ||\mathbf{w}' - \mathbf{e}_1||$ for all $\mathbf{w}' \in W$).

   - Consider the vector $\mathbf{w}_n = (1, -1/n, -1/n, \dots, -1/n, 0, 0, \dots)$ with first entry 1 followed by n entries equal to $-1/n$, and other entries 0.
   - Then $||\mathbf{w}_n - \mathbf{e}_1|| = ||(0, -1/n, -1/n, \dots, -1/n, 0, 0, \dots)|| = n \cdot (1/n)^2 = 1/n$ as desired.
   - Since $1/n \to 0$ as $n \to \infty$, a best approximation vector $\mathbf{w}$ would necessarily have $||\mathbf{w} - \mathbf{e}_1|| = 0$, but this is impossible since $\mathbf{e}_1 \notin W$.

   (e) Let $T : V \to V$ be the linear transformation defined by setting $T(\mathbf{e}_n) = \sum_{i=1}^{n} \mathbf{e}_i$ for each $i \geq 1$. If T had an adjoint $T^* : V \to V$, show that infinitely many components of $T^*(\mathbf{e}_1)$ would be nonzero. Deduce that T^* cannot exist.

   - By hypothesis we have $\langle \mathbf{e}_k, T^*(\mathbf{e}_1) \rangle = \langle T(\mathbf{e}_k), \mathbf{e}_1 \rangle = \langle \sum_{i=1}^{n} \mathbf{e}_i, \mathbf{e}_1 \rangle = 1$ for each $k \geq 1$.
   - Conjugating then yields $\langle T^*(\mathbf{e}_1), \mathbf{e}_k \rangle = 1$: but this inner product is the coefficient of $\mathbf{e}_k$ in $T^*(\mathbf{e}_1)$, so we would necessarily have $T^*(\mathbf{e}_1) = \sum_{k=1}^{\infty} \mathbf{e}_k$. But this vector is not an element of V since it has infinitely many nonzero components. This is a contradiction, so T^* cannot exist.
