E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 8, due Fri Mar 21st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If $T(\mathbf{v}) = \lambda \mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.
   (b) Every linear transformation on V has at least one eigenvector.
   (c) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.
   (d) Any two eigenvectors of T are linearly independent.
   (e) The sum of two eigenvectors of T is also an eigenvector of T.
   (f) The sum of two eigenvalues of T is also an eigenvalue of T.
   (g) If two matrices are similar, then they have the same eigenvectors.
   (h) If two matrices have the same eigenvalues, then they are similar.
   (i) If two matrices are similar, then they have the same eigenvalues.
   (j) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.
   (k) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.
   (l) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.
   (m) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.
   (n) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

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2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable over F and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

   (a) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.
   (b) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.
   (c) The matrix $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.
   (d) The matrix $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.
   (e) The matrix $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.
   (f) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

   (a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.
   (b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.
   (c) The transpose map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(M) = M^T$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let $V = C[0, 2\pi]$ with inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$. Also define $\varphi_0(x) = \dfrac{1}{\sqrt{2\pi}}$, and for positive integers k set $\varphi_{2k-1}(x) = \dfrac{1}{\sqrt{\pi}}\cos(kx)$ and $\varphi_{2k}(x) = \dfrac{1}{\sqrt{\pi}}\sin(kx)$.

   (a) Show that $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ is an orthonormal set in V. [Hint: Use the product-to-sum identities.]

   (b) Let $f(x) = x$. Find $||f||$ and $\langle f, \varphi_n\rangle$ for each $n \geq 0$. (You don't need to give details of the integral evaluations, just the resulting values.)

   (c) With $f(x) = x$, assuming that $f(x) = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle\, \varphi_k(x)$, derive Leibniz's formula $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$. [Hint: Set $x = \pi/2$.]

   (d) With $f(x) = x$, assuming that $||f||^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle^2$ (see problem 10 of the midterm for why this is a reasonable statement), find the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

   Remarks: The identity $||f||^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle^2$ is known as Parseval's identity. The problem of computing the value of the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known as the Basel problem. The correct value was (famously) first identified by Euler, who evaluated the sum by decomposing the function $\frac{\sin \pi x}{\pi x}$ as the infinite product $\prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2})$ and then comparing the power series coefficients of both sides.

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5. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Find all of the eigenvalues and a basis for each eigenspace of L.

   (b) Find all of the eigenvalues and a basis for each eigenspace of R.

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6. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.

   (a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.

   (b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

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7. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

   (a) If $B = -\dfrac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

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8. [Challenge] The goal of this problem is to give some counterexamples for results about orthogonal complements, projections, best approximations, and adjoints in infinite-dimensional spaces. Let V be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ with only finitely many nonzero terms, with inner product given by $\langle \{a_i\}, \{b_i\}\rangle = \sum_{i=1}^{\infty} a_i b_i$. (Note that this sum converges since only finitely many terms are nonzero.) Let $\mathbf{e}_i$ be the ith unit coordinate vector and observe that $\{\mathbf{e}_i\}_{i\geq 1}$ is an orthonormal basis for V. Now for each $n \geq 2$, let $\mathbf{v}_n = \mathbf{e}_1 - \mathbf{e}_n$ and define $W = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \dots)$.

   (a) Show that $\mathbf{e}_1 \notin W$ so that W is a proper subspace of V, but that $W^\perp = \{\mathbf{0}\}$.

   (b) Show that $W^\perp + W \neq V$ and that $(W^\perp)^\perp \neq W$.

   (c) For any $\mathbf{v} \notin W$, show that there does not exist any choice of $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. Conclude that there is not a well-defined orthogonal projection map of V onto W.

   (d) Show that there exists a vector $\mathbf{w}_n \in W$ such that $||\mathbf{w}_n - \mathbf{e}_1|| = 1/n$ for any positive integer n. Deduce that there is no possible best approximation vector $\mathbf{w}$ to $\mathbf{e}_1$ inside W (namely with $||\mathbf{w} - \mathbf{e}_1|| \leq ||\mathbf{w}' - \mathbf{e}_1||$ for all $\mathbf{w}' \in W$).

   (e) Let $T : V \to V$ be the linear transformation defined by setting $T(\mathbf{e}_n) = \sum_{i=1}^{n} \mathbf{e}_i$ for each $i \geq 1$. If T had an adjoint $T^* : V \to V$, show that infinitely many components of $T^*(\mathbf{e}_1)$ would be nonzero. Deduce that T^* cannot exist.
