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1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F with $\mathbf{v}, \mathbf{w} \in V$, and let W be a subspace of V. Identify each of the following statements as true or false:

   (a) An orthogonal set of vectors is linearly independent.

   - False: the set $\{(0,0,0),(1,0,0)\}$ is orthogonal but not linearly independent.

   (b) An orthonormal set of vectors is linearly independent.

   - True: we showed any set of nonzero orthogonal vectors is linearly independent, and an orthonormal set is orthogonal and cannot include the zero vector (since its norm is 0).

   (c) Every finite-dimensional inner product space has an orthonormal basis.

   - True: we can construct an orthonormal basis via Gram-Schmidt.

   (d) If V is finite-dimensional and W is any subspace of V, then $\dim(W) = \dim(W^\perp)$.

   - False: the correct formula is $\dim(W) + \dim(W^\perp) = \dim(V)$.

   (e) If $\mathbf{w}^\perp$ is a vector in $W^\perp$, then the orthogonal projection of $\mathbf{w}^\perp$ onto W is $\mathbf{w}^\perp$ itself.

   - False: the orthogonal projection of a vector $\mathbf{w}^\perp$ in $W^\perp$ onto W is zero.

   (f) If $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthonormal basis of W, then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of $\mathbf{v}$ into W.

   - True: this is the orthogonal projection formula we proved.

   (g) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V, the vector $\mathbf{w} \in W$ minimizing $||\mathbf{v} - \mathbf{w}||$ is the orthogonal projection of $\mathbf{v}$ into W.

   - True: this is the best-approximation property of the orthogonal projection.

   (h) If $T : V \to V$ is linear, then the adjoint of T exists and is unique.

   - False: the adjoint does not always necessarily exist over an arbitrary vector space. (If it does exist, then it is unique.)

   (i) If $T : V \to V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.

   - True: we proved that the adjoint always exists over finite-dimensional vector spaces.

   (j) If $T : V \to F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.

   - True: this is the version of the Riesz representation theorem we established.

   (k) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.

   - False: the correct formula is $(S + iT)^* = S^* - iT^*$.

   (l) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(ST)^* = S^*T^*$.

   - False: the correct formula is $(ST)^* = T^*S^*$.

   (m) If $A\mathbf{x} = \mathbf{c}$ is an inconsistent system of linear equations, then the best approximation of a solution is given by the solutions $\hat{\mathbf{x}}$ of $A^*\hat{\mathbf{x}} = A^*A\mathbf{c}$.

   - False: the correct equation to solve is the normal equation $(A^*A)\hat{\mathbf{x}} = A^*\mathbf{c}$.

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2. Calculate the following things:

(a) The result of applying Gram-Schmidt to the vectors $\mathbf{v}_1 = (1, 2, 0, -2)$, $\mathbf{v}_2 = (1, -1, 4, 4)$, $\mathbf{v}_3 = (6, 6, 0, -9)$ in $\mathbb{R}^4$ under the dot product.

- First, $\mathbf{w}_1 = \mathbf{v}_1 = \boxed{(1, 2, 0, -2)}$.
- Next, $\mathbf{w}_2 = \mathbf{v}_2 - a_1\mathbf{w}_1$, where $a_1 = \dfrac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(1, -1, 4, 4) \cdot (1, 2, 0, -2)}{(1, 2, 0, -2) \cdot (1, 2, 0, -2)} = \dfrac{-9}{9} = -1$. Thus, $\mathbf{w}_2 = (1, -1, 4, 4) + (1, 2, 0, -2) = \boxed{(2, 1, 4, 2)}$.
- Finally, $\mathbf{w}_3 = \mathbf{v}_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} = \dfrac{(6, 6, 0, -9) \cdot (1, 2, 0, -2)}{(1, 2, 0, -2) \cdot (1, 2, 0, -2)} = 4$, and $b_2 = \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} = \dfrac{(6, 6, 0, -9) \cdot (2, 1, 4, 2)}{(2, 1, 4, 2) \cdot (2, 1, 4, 2)} = 0$. Thus, $\mathbf{w}_3 = (6, 6, 0, -9) - 4(1, 2, 0, -2) - 0(2, 1, 4, 2) = \boxed{(2, -2, 0, -1)}$.

(b) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside $\mathbb{R}^4$ under the dot product.

- The orthogonal complement corresponds to the nullspace of the matrix whose rows are the given vectors.
- Row-reducing $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \end{bmatrix}$ yields the reduced row-echelon form $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$.
- From the reduced row-echelon form, we see that $\boxed{\{(-2, 1, 0, 1), (1, -2, 1, 0)\}}$ is a basis for the nullspace and hence of $W^\perp$.

(c) The orthogonal decomposition $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside $\mathbb{R}^3$ under the dot product. Also, verify the relation $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.

- Notice that the vectors $\mathbf{e}_1 = \frac{1}{3}(1, 2, 2)$ and $\mathbf{e}_2 = \frac{1}{3}(2, -2, 1)$ form an orthonormal basis for W.
- Thus, the orthogonal projection is $\mathbf{w} = \text{proj}_W(\mathbf{v}) = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{v}, \mathbf{e}_2 \rangle \mathbf{e}_2 = 8\mathbf{e}_1 + 5\mathbf{e}_2 = \boxed{(6, 2, 7)}$.
- We see that $\mathbf{w}^\perp = \mathbf{v} - \mathbf{w} = (-4, -2, 4)$ is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$ so it is indeed in $W^\perp$.
- Furthermore, $||\mathbf{v}||^2 = 125$, while $||\mathbf{w}||^2 = 89$ and $||\mathbf{w}^\perp||^2 = 36$, so indeed $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.

(d) An orthogonal basis for $W = \text{span}[x, x^2, x^3]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

- We start with $\mathbf{w}_1 = p_1 = \boxed{x}$.
- Next, $\mathbf{w}_2 = p_2 - a_1\mathbf{w}_1$, where $a_1 = \dfrac{\langle p_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \dfrac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} = 0$. Thus, $\mathbf{w}_2 = \boxed{x^2}$.
- Finally, $\mathbf{w}_3 = p_3 - b_1\mathbf{w}_1 - b_2\mathbf{w}_2$ where $b_1 = \dfrac{\langle p_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} = \dfrac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} = \dfrac{3}{5}$, and $b_2 = \dfrac{\langle p_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} = \dfrac{\int_{-1}^{1} x^5\,dx}{\int_{-1}^{1} x^4\,dx} = 0$. Thus, $\mathbf{w}_3 = \boxed{x^3 - \dfrac{3}{5}x}$.

(e) The orthogonal projection of $\mathbf{v} = 1+2x^2$ into $\text{span}[x, x^2, x^3]$, with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

- Using the basis from (d) we see $\mathbf{w} = \dfrac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle}\mathbf{e}_1 + \dfrac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle}\mathbf{e}_2 + \dfrac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle}\mathbf{e}_3 = 0\mathbf{e}_1 - \dfrac{11}{3}\mathbf{e}_2 + 0\mathbf{e}_3 = \boxed{\dfrac{11}{3}x^2}$.

(f) The quadratic polynomial $p(x) \in P_2(\mathbb{R})$ that minimizes the expression $\int_0^1 [p(x) - \sqrt{x}]^2\,dx$.

- The desired polynomial is the orthogonal projection of $f(x) = \sqrt{x}$ into $W = P_2(\mathbb{R})$. Using Gram-Schmidt we can find an orthogonal basis for W, which yields $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = -1+2x$, $\mathbf{e}_3 = 1-6x+6x^2$.
- Using the orthogonal basis, we get the projection $\mathbf{w} = \dfrac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle}\mathbf{e}_1 + \dfrac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle}\mathbf{e}_2 + \dfrac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle}\mathbf{e}_3 = \dfrac{2}{3}\mathbf{e}_1 + \dfrac{2}{5}\mathbf{e}_2 - \dfrac{2}{21}\mathbf{e}_3 = \boxed{\dfrac{6}{35} + \dfrac{48}{35}x - \dfrac{4}{7}x^2}$.

(g) The least-squares solution to the inconsistent system $x + 3y = 9$, $3x + y = 5$, $x + y = 2$.

- We have $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}$. Since A clearly has rank 2, $A^T A$ will be invertible and there will be a unique least-squares solution.
- We compute $A^*A = \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}$, which is indeed invertible and has inverse $(A^*A)^{-1} = \frac{1}{72}\begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix}$.
- The least-squares solution is therefore $\hat{\mathbf{x}} = (A^*A)^{-1}A^*\mathbf{c} = \boxed{\begin{bmatrix} 2/3 \\ 8/3 \end{bmatrix}}$.

(h) The least-squares line $y = a + bx$ approximating the points $\{(4, 7), (11, 21), (15, 29), (19, 35), (30, 49)\}$. (Give three decimal places.)

- We seek the least-squares solution for $A\mathbf{x} = \mathbf{c}$, where $A = \begin{bmatrix} 1 & 4 \\ 1 & 11 \\ 1 & 15 \\ 1 & 19 \\ 1 & 30 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 7 \\ 21 \\ 29 \\ 35 \\ 49 \end{bmatrix}$.
- We compute $A^*A = \begin{bmatrix} 5 & 79 \\ 79 & 1623 \end{bmatrix}$, so the least-squares solution is $\hat{\mathbf{x}} = (A^*A)^{-1}A^*\mathbf{c} \approx \begin{bmatrix} 2.856 \\ 1.604 \end{bmatrix}$.
- Thus, to three decimal places, the desired line is $y = \boxed{1.604x + 2.856}$.

(i) The least-squares quadratic $y = a + bx + cx^2$ approximating the points $\{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\}$. (Give three decimal places.)

- We seek the least-squares solution for $A\mathbf{x} = \mathbf{c}$, with $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 22 \\ 11 \\ 4 \\ 3 \\ 13 \end{bmatrix}$.
- We compute $A^*A = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$, so the least-squares solution is $\hat{\mathbf{x}} = (A^*A)^{-1}A^*\mathbf{c} \approx \begin{bmatrix} 3.743 \\ -2.6 \\ 3.429 \end{bmatrix}$.
- Thus, the desired quadratic polynomial is $y = \boxed{3.743 - 2.6x + 3.429x^2}$.

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3. Let V be an inner product space with scalar field F. The goal of this problem is to prove the so-called "polarization identities".

(a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4}||\mathbf{v} - \mathbf{w}||^2$.

- We just expand the norms on the right-hand side: $||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle - \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle = [\langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] - [\langle \mathbf{v}, \mathbf{v} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] = 4\langle \mathbf{v}, \mathbf{w} \rangle$ so dividing by 4 yields the claimed result.

(b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}||\mathbf{v} + \mathbf{w}||^2 + \frac{i}{4}||\mathbf{v} + i\mathbf{w}||^2 - \frac{1}{4}||\mathbf{v} - \mathbf{w}||^2 - \frac{i}{4}||\mathbf{v} - i\mathbf{w}||^2$.

- As in part (a) we just expand the norms on the right-hand side:

$$\begin{aligned}
\sum_{k=1}^{4} i^k \left|\left|\mathbf{v} + i^k\mathbf{w}\right|\right|^2 &= ||\mathbf{v} + \mathbf{w}||^2 + i\,||\mathbf{v} + i\mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2 - i\,||\mathbf{v} - i\mathbf{w}||^2 \\
&= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + i\langle \mathbf{v} + i\mathbf{w}, \mathbf{v} + i\mathbf{w} \rangle - \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle - i\langle \mathbf{v} - i\mathbf{w}, \mathbf{v} - i\mathbf{w} \rangle \\
&= [\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] + i[\langle \mathbf{v}, \mathbf{v} \rangle - i\langle \mathbf{v}, \mathbf{w} \rangle + i\langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] \\
&\quad - [\langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] - i[\langle \mathbf{v}, \mathbf{v} \rangle + i\langle \mathbf{v}, \mathbf{w} \rangle - i\langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle] \\
&= 4\langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

and so dividing by 4 yields the claimed result.

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4. Let V be a finite-dimensional inner product space and W be a subspace of V.

(a) Prove that $W \cap W^\perp = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^\perp$. [Hint: Use $\dim(W) + \dim(W^\perp) = \dim(V)$.]

- Suppose $\mathbf{w} \in W \cap W^\perp$. Then $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ since the inner product of any vector in W with any vector in $W^\perp$ is 0. But then property [I3] of the inner product immediately implies $\mathbf{w} = \mathbf{0}$, so $W \cap W^\perp = \{\mathbf{0}\}$.
- For the second statement, per the hint, since $\dim(W) + \dim(W^\perp) = \dim(V)$ and $W \cap W^\perp = \{\mathbf{0}\}$ we must have $W + W^\perp = V$ (in fact we already proved this by showing that the union of a basis of W and a basis of $W^\perp$ gives a basis for V). Hence by the definition of direct sum, we have $V = W \oplus W^\perp$.

(b) Let $T : V \to W$ be the function defined by setting $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ for $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$. Prove that T is linear, that $T^2 = T$, that $\text{im}(T) = W$, and that $\ker(T) = W^\perp$. Conclude that T is projection onto the subspace W with kernel $W^\perp$.

- Suppose $\mathbf{v}_1 = \mathbf{w}_1 + \mathbf{w}_1^\perp$ and $\mathbf{v}_2 = \mathbf{w}_2 + \mathbf{w}_2^\perp$ where the $\mathbf{w}_i \in W$ and the $\mathbf{w}_i^\perp \in W^\perp$.
- Then for any scalar c, we see $\mathbf{v}_1 + c\mathbf{v}_2 = (\mathbf{w}_1 + c\mathbf{w}_2) + (\mathbf{w}_1^\perp + c\mathbf{w}_2^\perp)$ where $\mathbf{w}_1 + c\mathbf{w}_2 \in W$ and $\mathbf{w}_1^\perp + c\mathbf{w}_2^\perp \in W^\perp$ since these are both subspaces. Thus, by uniqueness of orthogonal decomposition, this is the orthogonal decomposition of $\mathbf{v}_1 + c\mathbf{v}_2$.
- Then $T(\mathbf{v}_1 + c\mathbf{v}_2) = \mathbf{w}_1 + c\mathbf{w}_2 = T(\mathbf{v}_1) + cT(\mathbf{v}_2)$ so T is linear.
- Also, $T^2(\mathbf{v}) = T(T(\mathbf{v})) = T(\mathbf{w}) = \mathbf{w}$ since we can clearly write $\mathbf{w} = \mathbf{w} + \mathbf{0}$ so $T(\mathbf{w}) = \mathbf{w}$ again by uniqueness. Hence $T^2(\mathbf{v}) = T(\mathbf{v})$ for every $\mathbf{v} \in V$, so T is a projection map.

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5. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. We say T is <u>Hermitian</u> (or <u>self-adjoint</u>) if $T = T^*$, and that T is <u>skew-Hermitian</u> if $T = -T^*$.

(a) Show that T is Hermitian if and only if iT is skew-Hermitian.

- Note $(iT)^* = -iT^*$ so $T = T^*$ if and only if $(iT)^* = -iT$.

(b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.

- Note $(T + T^*)^* = T^* + T^{**} = T + T^*$, $(T^*T)^* = T^*T^{**} = T^*T$, and $(TT^*)^* = T^{**}T^* = TT^*$.
- Also, $(T - T^*)^* = T^* - T^{**} = T^* - T$.

(c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.

- In such a case we would necessarily have $T^* = (S_1 + iS_2)^* = S_1^* - iS_2^* = S_1 - iS_2$.
- Solving for S_1 and S_2 in terms of T and T^* then yields $S_1 = \dfrac{1}{2}(T + T^*)$ and $S_2 = \dfrac{1}{2i}(T - T^*)$, so these are the only possible choices.
- On the other hand, by (a) and (b), we see that these S_1 and S_2 are in fact Hermitian, so these are the unique choices.

(d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector $\mathbf{v}$.

- If $T^* = T$ then $\langle T\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, T^*\mathbf{v} \rangle = \langle \mathbf{v}, T\mathbf{v} \rangle = \overline{\langle T\mathbf{v}, \mathbf{v} \rangle}$, so $\langle T\mathbf{v}, \mathbf{v} \rangle$ equals its conjugate hence is real.

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6. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$) and $T : V \to V$ is linear. We say T is a "distance-preserving map" on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is a "pairing-preserving map" on V if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

(a) Prove that T is distance-preserving if and only if it is pairing-preserving. [Hint: Use problem 3.]

- Clearly if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle$ then $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle T\mathbf{v}, T\mathbf{v} \rangle = ||T\mathbf{v}||^2$.
- The converse follows by using the polarization identities from problem 4 to recover the inner product from the norm.
- For example, if $F = \mathbb{R}$ we have $\langle T\mathbf{v}, T\mathbf{w} \rangle = \dfrac{1}{4} ||T(\mathbf{v} + \mathbf{w})||^2 - \dfrac{1}{4} ||T(\mathbf{v} - \mathbf{w})||^2 = \dfrac{1}{4} ||\mathbf{v} + \mathbf{w}||^2 - \dfrac{1}{4} ||\mathbf{v} - \mathbf{w}||^2 = \langle \mathbf{v}, \mathbf{w} \rangle$, and similarly in the complex case.

A map $T : V \to V$ satisfying the distance- and pairing-preserving conditions is called a (linear) isometry.

(b) Show that the transformations $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.

- We simply compute $||S(x, y, z)|| = z^2 + x^2 + y^2 = ||(x, y, z)||$, and $||T(x, y, z)|| = \dfrac{1}{9}[(x + 2y + 2z)^2 + (2x + y - 2z)^2 + (2x - 2y + z)^2] = x^2 + y^2 + z^2 = ||(x, y, z)||$.
- Thus, S and T both preserve norms, so by the above, they are isometries.

(c) Show that isometries are one-to-one.

- Suppose T is an isometry. If $\mathbf{v} \in \ker(T)$ then $||\mathbf{v}|| = ||T(\mathbf{v})|| = 0$, so $\mathbf{v} = 0$ by [I3]. Thus T is one-to-one.

(d) Show that isometries preserve orthogonal and orthonormal sets.

- Suppose T is an isometry. If $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ then $\langle T\mathbf{v}, T\mathbf{w} \rangle = 0$.
- Thus, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ is an orthogonal set, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots\}$ is also orthogonal.
- Furthermore, since $||\mathbf{v}|| = ||T(\mathbf{v})||$, T also preserves orthonormal sets.

(e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.

- Observe that $\langle \mathbf{v}, \mathbf{w} \rangle - \langle T\mathbf{v}, T\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, T^*T\mathbf{w} \rangle = \langle \mathbf{v}, (I - T^*T)\mathbf{w} \rangle$.
- Thus, setting $\mathbf{v} = (I - T^*T)\mathbf{w}$ shows that the left-hand side is identically zero if and only if $(I - T^*T)\mathbf{w}$ is identically zero, if and only if $T^*T = I$.

(f) We say that a matrix $A \in M_{n\times n}(F)$ is unitary if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

- This is simply the matrix version of (e): the isometries are the linear transformations with $A^*A = I_n$, and $A^*A = I_n$ is equivalent to saying that $A^{-1} = A^*$.

Remark: Notice that $A \in M_{n\times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of $\mathbb{C}$. Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on $\mathbb{C}^n$ (or $\mathbb{R}^n$) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).

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7. [Challenge] The goal of this problem is to give an example of an inner product space that has no orthonormal basis. Let $V = \ell^2(\mathbb{R})$ be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ such that $\sum_{i=1}^{\infty} a_i^2$ is finite, under componentwise addition and scalar multiplication.

(a) Show that the pairing $\langle \{a_i\}_{i\geq 1}, \{b_i\}_{i\geq 1} \rangle = \sum_{i=1}^{\infty} a_i b_i$ is an inner product on V. (Make sure to justify why this sum converges.)

- First, we must justify why the pairing is well defined: starting with the Cauchy-Schwarz inequality $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2$, if we take the limit as $n \to \infty$, then we obtain $\left(\sum_{i=1}^{\infty} a_i b_i\right)^2 \leq \sum_{i=1}^{\infty} a_i^2 \cdot \sum_{i=1}^{\infty} b_i^2$. But the right-hand side is finite by the assumption that $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ are elements of V, and so this means $|\sum_{i=1}^{\infty} a_i b_i|$ is finite (i.e., the sum converges).
- Now we just verify the three parts of the definition of an inner product. Let $\mathbf{v} = \{a_i\}_{i\geq 1}$, $\mathbf{v}' = \{a_i'\}_{i\geq 1}$, $\mathbf{w} = \{b_i\}_{i\geq 1}$.
- [I1]: We have $\langle \mathbf{v} + \alpha\mathbf{v}', \mathbf{w} \rangle = \sum_{i=1}^{\infty} (a_i + \alpha a_i')b_i = \sum_{i=1}^{\infty} a_i b_i + \alpha \sum_{i=1}^{\infty} a_i' b_i = \langle \mathbf{v}, \mathbf{w} \rangle + \alpha \langle \mathbf{v}', \mathbf{w} \rangle$.
- [I2]: We have $\langle \mathbf{w}, \mathbf{v} \rangle = \sum_{i=1}^{\infty} b_i a_i = \sum_{i=1}^{\infty} a_i b_i = \langle \mathbf{v}, \mathbf{w} \rangle$.
- [I3]: We have $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{\infty} a_i^2$ which is a sum of squares hence always nonnegative, and it is zero only when all $a_i = 0$, which is to say, when $\mathbf{v} = \mathbf{0}$.

(b) Let $\mathbf{v}_i \in V$ be the sequence with a 1 in the ith component and 0s elsewhere. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$ is an orthonormal set in V and that the only vector $\mathbf{w}$ orthogonal to all of the $\mathbf{v}_i$ is the zero vector. Deduce that S is a maximal orthonormal set of V that is not a basis of V.

- It is obvious that S is orthonormal, since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$ and $||\mathbf{v}_i|| = 1$ for all i.
- If $\mathbf{w} = \{b_i\}_{i \geq 1}$, then $\langle \mathbf{w}, \mathbf{v}_i \rangle = b_i$ since the only term that survives in the sum is $b_i \cdot 1$ in the ith term. So if $\langle \mathbf{w}, \mathbf{v}_i \rangle = 0$ for all i, then each $b_i = 0$ so that $\mathbf{w} = 0$.
- This means that S is a maximal orthonormal subset of V, since it is orthonormal and there are no vectors that could be added to it to preserve orthonormality (since the only vector orthogonal to all of S is the zero vector, by the above).
- On the other hand, S is not a basis of V, since for example the vector $\{1/2^i\}_{i \geq 1}$, which is in V since $\sum_{i=1}^{\infty} (1/2^i)^2 = 1/3$ is finite, cannot be written as a finite linear combination of the vectors in S.

Part (b) shows that Gram-Schmidt does not necessarily construct an orthonormal basis of V. In fact, V has no orthonormal basis at all.

(c) Suppose V has an orthonormal basis $\{\mathbf{e}_i\}_{i \in I}$ for some indexing set I (which is necessarily infinite), and choose a countably infinite subset $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \dots$. Show that the sum $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$ is a well-defined vector in V that cannot be written as a (finite) linear combination of the basis $\{\mathbf{e}_i\}_{i \in I}$. [Hint: Show that $||\mathbf{v}||^2 = \lim_{n \to \infty} \left|\left| \sum_{k=1}^{n} 2^{-k} \mathbf{e}_k \right|\right|^2$ is finite.]

- We need to show that the sum for $\mathbf{v}$ converges to a vector in V. Clearly, since the vectors $\mathbf{e}_i$ in the sum all have length 1, each of their coordinates is at most 1, so the ith coordinate of $\sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$ is at most $\sum_{k=1}^{\infty} 2^{-k} = 1$. So the coordinates of the sum certainly all converge to a finite value.
- Furthermore, the sum is in fact in V: since the vectors $\mathbf{e}_i$ are orthonormal, we have $||\mathbf{v}||^2 = \lim_{n \to \infty} \left|\left| \sum_{k=1}^{n} 2^{-k} \mathbf{e}_k \right|\right|^2 = \lim_{n \to \infty} \sum_{k=1}^{n} 2^{-2k} = \lim_{n \to \infty} \frac{1}{3}(1 - 1/2^{-2n}) = \frac{1}{3}$, so in fact $||\mathbf{v}||^2$ is finite and so $\mathbf{v}$ is in V.
- However, we claim that $\mathbf{v}$ cannot be written as a finite linear combination of the basis elements $\mathbf{e}_i$.
- To see this, first note that since $\{\mathbf{e}_i\}_{i \in I}$ is an orthonormal basis of V, the the inner product $\langle \mathbf{v}, \mathbf{e}_i \rangle$ gives the coefficient of $\mathbf{e}_i$ in the decomposition of $\mathbf{v}$. But since $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k} \mathbf{e}_k$, we see that $\langle \mathbf{v}, \mathbf{e}_k \rangle = 2^{-k}$ is nonzero for infinitely many of the basis vectors $\mathbf{e}_k$. This precludes the possibility that $\mathbf{v}$ is a finite linear combination of the basis elements $\mathbf{e}_i$, which is a contradiction since $\{\mathbf{e}_i\}_{i \in I}$ was assumed to be a basis.

Remark: The point here is that because our definition of span and basis only allows us to use finite linear combinations, these definitions are not well suited to handle infinite-dimensional spaces like $\ell^2(\mathbb{R})$. However, it is possible (by exploiting the fact that ℓ^2 is a topologically-complete metric space) to deal with these issues and define a "Schauder basis" that allows the use of infinite sums, which amounts to viewing ℓ^2 as a Hilbert space.
