E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 7, due Fri Mar 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F with $\mathbf{v}, \mathbf{w} \in V$, and let W be a subspace of V. Identify each of the following statements as true or false:

   (a) An orthogonal set of vectors is linearly independent.
   (b) An orthonormal set of vectors is linearly independent.
   (c) Every finite-dimensional inner product space has an orthonormal basis.
   (d) If V is finite-dimensional and W is any subspace of V, then $\dim(W) = \dim(W^\perp)$.
   (e) If $\mathbf{w}^\perp$ is a vector in $W^\perp$, then the orthogonal projection of $\mathbf{w}^\perp$ onto W is $\mathbf{w}^\perp$ itself.
   (f) If $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is an orthonormal basis of W, then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.
   (g) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V, the vector $\mathbf{w} \in W$ minimizing $||\mathbf{v} - \mathbf{w}||$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.
   (h) If $T : V \to V$ is linear, then the adjoint of T exists and is unique.
   (i) If $T : V \to V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.
   (j) If $T : V \to F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.
   (k) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.
   (l) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(ST)^* = S^*T^*$.
   (m) If $A\mathbf{x} = \mathbf{c}$ is an inconsistent system of linear equations, then the best approximation of a solution is given by the solutions $\hat{\mathbf{x}}$ of $A^*\hat{\mathbf{x}} = A^*A\mathbf{c}$.

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2. Calculate the following things:

   (a) The result of applying Gram-Schmidt to the vectors $\mathbf{v}_1 = (1, 2, 0, -2)$, $\mathbf{v}_2 = (1, -1, 4, 4)$, $\mathbf{v}_3 = (6, 6, 0, -9)$ in $\mathbb{R}^4$ under the dot product.
   (b) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside $\mathbb{R}^4$ under the dot product.
   (c) The orthogonal decomposition $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside $\mathbb{R}^3$ under the dot product. Also, verify the relation $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.
   (d) An orthogonal basis for $W = \text{span}[x, x^2, x^3]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.
   (e) The orthogonal projection of $\mathbf{v} = 1 + 2x^2$ into $\text{span}[x, x^2, x^3]$, with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.
   (f) The quadratic polynomial $p(x) \in P_2(\mathbb{R})$ that minimizes the expression $\int_0^1 [p(x) - \sqrt{x}]^2\,dx$.
   (g) The least-squares solution to the inconsistent system $x + 3y = 9$, $3x + y = 5$, $x + y = 2$.
   (h) The least-squares line $y = a + bx$ approximating the points $\{(4, 7), (11, 21), (15, 29), (19, 35), (30, 49)\}$. (Give three decimal places.)
   (i) The least-squares quadratic $y = a + bx + cx^2$ approximating the points $\{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\}$. (Give three decimal places.)

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Let V be an inner product space with scalar field F. The goal of this problem is to prove the so-called "polarization identities".

   (a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4}||\mathbf{v} - \mathbf{w}||^2$.
   (b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}||\mathbf{v} + \mathbf{w}||^2 + \frac{i}{4}||\mathbf{v} + i\mathbf{w}||^2 - \frac{1}{4}||\mathbf{v} - \mathbf{w}||^2 - \frac{i}{4}||\mathbf{v} - i\mathbf{w}||^2$.

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4. Let V be a finite-dimensional inner product space and W be a subspace of V.
   (a) Prove that $W \cap W^{\perp} = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^{\perp}$. [Hint: Use $\dim(W) + \dim(W^{\perp}) = \dim(V)$.]
   (b) Let $T : V \to W$ be the function defined by setting $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ for $\mathbf{w} \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$. Prove that T is linear, that $T^2 = T$, that $\mathrm{im}(T) = W$, and that $\ker(T) = W^{\perp}$. Conclude that T is projection onto the subspace W with kernel $W^{\perp}$.

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5. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is linear and possesses an adjoint T^*. We say T is Hermitian (or self-adjoint) if $T = T^*$, and that T is skew-Hermitian if $T = -T^*$.
   (a) Show that T is Hermitian if and only if iT is skew-Hermitian.
   (b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.
   (c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.
   (d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is a real number for any vector $\mathbf{v}$.

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6. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$) and $T : V \to V$ is linear. We say T is a "distance-preserving map" on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is a "pairing-preserving map" on V if $\langle \mathbf{v}, \mathbf{w}\rangle = \langle T\mathbf{v}, T\mathbf{w}\rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.
   (a) Prove that T is distance-preserving if and only if it is pairing-preserving. [Hint: Use problem 3.]

   A map $T : V \to V$ satisfying the distance- and pairing-preserving conditions is called a (linear) isometry.
   (b) Show that the transformations $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.
   (c) Show that isometries are one-to-one.
   (d) Show that isometries preserve orthogonal and orthonormal sets.
   (e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.
   (f) We say that a matrix $A \in M_{n\times n}(F)$ is unitary if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

   Remark: Notice that $A \in M_{n\times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of $\mathbb{C}$. Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on $\mathbb{C}^n$ (or $\mathbb{R}^n$) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).

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7. [Challenge] The goal of this problem is to give an example of an inner product space that has no orthonormal basis. Let $V = \ell^2(\mathbb{R})$ be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ such that $\sum_{i=1}^{\infty} a_i^2$ is finite, under componentwise addition and scalar multiplication.
   (a) Show that the pairing $\langle \{a_i\}_{i\geq 1}, \{b_i\}_{i\geq 1}\rangle = \sum_{i=1}^{\infty} a_i b_i$ is an inner product on V. (Make sure to justify why this sum converges.)
   (b) Let $\mathbf{v}_i \in V$ be the sequence with a 1 in the ith component and 0s elsewhere. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$ is an orthonormal set in V and that the only vector $\mathbf{w}$ orthogonal to all of the $\mathbf{v}_i$ is the zero vector. Deduce that S is a maximal orthonormal set of V that is not a basis of V.

   Part (b) shows that Gram-Schmidt does not necessarily construct an orthonormal basis of V. In fact, V has no orthonormal basis at all.
   (c) Suppose V has an orthonormal basis $\{\mathbf{e}_i\}_{i\in I}$ for some indexing set I (which is necessarily infinite), and choose a countably infinite subset $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \dots$. Show that the sum $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k}\mathbf{e}_k$ is a well-defined vector in V that cannot be written as a (finite) linear combination of the basis $\{\mathbf{e}_i\}_{i\in I}$. [Hint: Show that $||\mathbf{v}||^2 = \lim_{n\to\infty} \left|\left|\sum_{k=1}^{n} 2^{-k}\mathbf{e}_k\right|\right|^2$ is finite.]

   Remark: The point here is that because our definition of span and basis only allows us to use finite linear combinations, these definitions are not well suited to handle infinite-dimensional spaces like $\ell^2(\mathbb{R})$. However, it is possible (by exploiting the fact that ℓ^2 is a topologically-complete metric space) to deal with these issues and define a "Schauder basis" that allows the use of infinite sums, which amounts to viewing ℓ^2 as a Hilbert space.
