E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 6, due Fri Feb 21st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that V is finite-dimensional over the field F, the bases α, β, γ are ordered, T is linear, and $\langle \cdot, \cdot \rangle$ is an inner product on V. Identify each of the following statements as true or false:
   (a) If $T : V \to V$ is an isomorphism, then $[T]_\beta^\beta = Q[T]_\alpha^\alpha Q^{-1}$ where $Q = [T]_\alpha^\beta$.
   (b) For any $T : V \to V$, there always exists an invertible matrix Q such that $[T]_\beta^\beta = Q[T]_\alpha^\alpha Q^{-1}$.
   (c) For any $T : V \to V$, if $P = [I]_\beta^\gamma$, then it is true that $[T]_\gamma^\gamma = P[T]_\beta^\beta P^{-1}$.
   (d) An inner product is linear in each of its components.
   (e) There is exactly one inner product on $\mathbb{R}^n$.
   (f) In any inner product space, $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.
   (g) In any inner product space, $||\mathbf{v} + \mathbf{w}|| \geq ||\mathbf{v}|| + ||\mathbf{w}||$.
   (h) In any inner product space, if $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ then $\mathbf{v} = \mathbf{0}$.
   (i) In any inner product space, if $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{y} \rangle$ then $\mathbf{x} = \mathbf{y}$.
   (j) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \to F$ with $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ is linear.
   (k) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \to F$ with $T(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle$ is linear.
   (l) The Cauchy-Schwarz inequality holds in every inner product space.
   (m) The triangle inequality holds in real inner product spaces but not complex inner product spaces.

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2. Suppose $V = P_3(\mathbb{R})$, with standard basis $\beta = \{1, x, x^2, x^3\}$, and let $T : V \to V$ be the linear transformation with $T(1) = 1 - x + x^2 - x^3$, $T(x) = 2x - x^3$, and $T(x^2) = 3 + x - x^3$, and $T(x^3) = 1 - x^2$.
   (a) Find $[T]_\beta^\beta$.
   Now let γ be the ordered basis $\gamma = \{x^3, x^2, x + 1, x\}$.
   (b) Find the change-of-basis matrix $Q = [I]_\beta^\gamma$ and its inverse.
   (c) For $\mathbf{v} = 2 - x - 2x^2 + x^3$, compute $[\mathbf{v}]_\beta$, $[\mathbf{v}]_\gamma$, and verify that $[\mathbf{v}]_\gamma = Q[\mathbf{v}]_\beta$.
   (d) Find $[T]_\beta^\gamma$, $[T]_\gamma^\beta$, and $[T]_\gamma^\gamma$.

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3. For each of the following pairings, determine (with brief justification) whether or not it is an inner product on the given vector space:
   (a) The pairing $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $M_{2\times2}(\mathbb{R})$.
   (b) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{R}^2$.
   (c) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{C}^2$.
   (d) The pairing $\langle (a, b), (c, d) \rangle = ac$ on $\mathbb{R}^2$.
   (e) The pairing $\langle f, g \rangle = \int_0^1 f'(x)g(x)\,dx$ on $C[0, 1]$.

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4. For each pair of vectors $\mathbf{v}, \mathbf{w}$ in the given inner product space, compute $\langle \mathbf{v}, \mathbf{w} \rangle$, $||\mathbf{v}||$, $||\mathbf{w}||$, and $||\mathbf{v} + \mathbf{w}||$, and verify the Cauchy-Schwarz and triangle inequalities for $\mathbf{v}$ and $\mathbf{w}$:
   (a) $\mathbf{v} = (1, 2, 2, 4)$ and $\mathbf{w} = (4, 1, 4, 4)$ in $\mathbb{R}^4$ with the standard inner product.
   (b) $\mathbf{v} = (i, -i, 1 + i)$ and $\mathbf{w} = (2 - i, 4, -2i)$ in $\mathbb{C}^3$ with the standard inner product.
   (c) $\mathbf{v} = e^t$ and $\mathbf{w} = e^{2t}$ in $C[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that V is a finite-dimensional vector space and $T : V \to V$ is linear.

   (a) Suppose there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. Show that T is a projection map (i.e., that $T^2 = T$).

   (b) Conversely, suppose that T is a projection map. Show that there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. [Hint: As shown on homework 4, $V = \ker(T) \oplus \operatorname{im}(T)$; take β be a basis of $\ker(T)$ followed by a basis of $\operatorname{im}(T)$.]

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6. Let V be an inner product space.

   (a) If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on V, show that $\langle \cdot, \cdot \rangle_3 = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is also an inner product on V, where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = \langle \mathbf{v}, \mathbf{w} \rangle_1 + \langle \mathbf{v}, \mathbf{w} \rangle_2$.

   (b) If $\langle \cdot, \cdot \rangle_1$ is an inner product on V and c is a positive real number, show that $\langle \cdot, \cdot \rangle_3 = c \langle \cdot, \cdot \rangle_1$ is also an inner product on V, where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = c \langle \mathbf{v}, \mathbf{w} \rangle_1$.

   (c) Does the collection of inner products on V form a vector space under the natural addition and scalar multiplication described above? Explain why or why not.

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7. Prove the following inequalities:

   (a) Prove that $(a_1 + a_2 + \cdots + a_n) \cdot (\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \geq n^2$ for any positive real numbers $a_1, a_2, \ldots, a_n$, with equality if and only if all of the a_i are equal.

   (b) If a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 \leq 5$, show that $a + 2b + 3c + 4d \leq 5\sqrt{6}$.

   (c) Prove Nesbitt's inequality: for any positive real numbers a, b, c it is true that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$. [Hint: Apply Cauchy-Schwarz to $(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $(1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$.]

   (d) Prove the following generalization of Cauchy-Schwarz: if $\langle \cdot, \cdot \rangle$ is an inner product on the vector space V then $\left[\sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{w}_j \rangle\right]^2 \leq \left[\sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{v}_j \rangle\right] \cdot \left[\sum_{j=1}^n \langle \mathbf{w}_j, \mathbf{w}_j \rangle\right]$ for any vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ in V. [Hint: Apply Cauchy-Schwarz to the inner product space $\tilde{V}$ of n-tuples of elements of V.]

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8. [Challenge] Let N be a positive integer and suppose we are given a set S_P of N points and a set S_L of N lines in the Euclidean plane. An "incidence" is defined to be a pair (P, L) with $P \in S_P$, $L \in S_L$, and where the point P lies on the line L. If I represents the total number of incidences, we have an obvious estimate $I \leq N^2$; the goal of this problem is to prove a substantially better estimate of $I \leq N^{3/2} + N$.

   (a) Show that $I = \sum_{P \in S_P} \sum_{L \in S_L} \delta_{P,L}$ where $\delta_{P,L} = 1$ if P lies on L and 0 otherwise.

   (b) For any fixed lines $L_1 \neq L_2$, show that $\sum_{P \in S_P} \delta_{P,L_1} \delta_{P,L_2} \leq 1$.

   (c) Show that $\sum_{P \in S_P} (\sum_{L \in S_L} \delta_{P,L})^2 \leq I + (N^2 - N)$. [Hint: Write $(\sum_{L \in S_L} \delta_{P,L})^2 = (\sum_{L_1 \in S_L} \delta_{P,L_1})(\sum_{L_2 \in S_L} \delta_{P,L_2})$ and then split apart into the terms where $L_1 = L_2$ and where $L_1 \neq L_2$.]

   (d) Show that $I^2 \leq IN + N(N^2 - N)$ and deduce that $I \leq N^{3/2} + N$. [Hint: Use Cauchy-Schwarz on the outer sum in (a).]

   Remark: This technique as used in algebraic combinatorics is often called the "L^2 method", and there are many open questions related to this one seeking optimal estimates and constructions for point-line incidences.
