1. Assume that the vector spaces U, V, W are finite-dimensional over the field F, the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that S, T are linear transformations. Identify each of the following statements as true or false:

   (a) The space $\mathcal{L}(V, W)$ of all linear transformations from V to W has dimension $\dim V \cdot \dim W$.

   - $\boxed{\text{True}}$: since $\mathcal{L}(V, W)$ is isomorphic to $M_{\dim(V)\times\dim(W)}(F)$, their dimensions are also equal.

   (b) If A is an $m \times n$ matrix of rank r, then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension r.

   - $\boxed{\text{False}}$: the solution space is the nullspace, which has dimension $n - r$ if the matrix has rank r.

   (c) If A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $\text{rank}(A) < n$.

   - $\boxed{\text{True}}$: in this case the nullspace must have dimension greater than 0, so by the nullity-rank theorem that means the rank must be less than n.

   (d) If A is an $n \times n$ matrix of rank n, then the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.

   - $\boxed{\text{True}}$: by the nullity-rank theorem, this means that the nullspace of A has dimension 0.

   (e) If the columns of A are all scalar multiples of some vector $\mathbf{v}$, then $\text{rank}(A) \le 1$.

   - $\boxed{\text{True}}$: the column space is spanned by $\mathbf{v}$ so its dimension (which is the rank of A) is at most 1.

   (f) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ is an element of $M_{m\times n}(F)$.

   - $\boxed{\text{False}}$: if $\dim(V) = m$ and $\dim(W) = n$ then $[T]_\beta^\gamma$ is an $n \times m$ matrix. (Try it!)

   (g) If $[S]_\alpha^\beta = [T]_\alpha^\beta$ then $S = T$.

   - $\boxed{\text{True}}$: the map associating a linear transformation with its associated matrix is an isomorphism, so two linear transformations have the same associated matrix if and only if they are equal.

   (h) If $[T]_\alpha^\beta = [T]_\gamma^\delta$ then $\alpha = \gamma$ and $\beta = \delta$.

   - $\boxed{\text{False}}$: for example if T is the identity map on $\mathbb{R}^2$ and α, β are the standard basis and γ, δ are twice the standard basis, the associated matrices are both the identity matrix but the bases are different.

   (i) If $S : V \to W$ and $T : V \to W$ then $[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$.

   - $\boxed{\text{True}}$: this is the correct rule for computing the matrix associated to a sum.

   (j) If $T : V \to W$ and $\mathbf{v} \in V$, then $[T]_\alpha^\beta[\mathbf{v}]_\beta = [T\mathbf{v}]_\alpha$.

   - $\boxed{\text{False}}$: the correct formula is $[T]_\alpha^\beta[\mathbf{v}]_\alpha = [T\mathbf{v}]_\beta$.

   (k) If $S : V \to W$ and $T : U \to V$, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma[T]_\alpha^\beta$.

   - $\boxed{\text{True}}$: this is the correct rule for computing the matrix associated to a composition.

   (l) If $T : V \to V$ has an inverse T^{-1}, then $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

   - $\boxed{\text{False}}$: the correct formula is $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$, since $[T^{-1}]_\gamma^\beta[T]_\beta^\gamma = [I]_\beta^\beta = I_n$.

   (m) If $T : V \to V$ has an inverse T^{-1}, then for any $\mathbf{v} \in V$, $[T^{-1}\mathbf{v}]_\gamma = ([T]_\gamma^\beta)^{-1}[\mathbf{v}]_\beta$.

   - $\boxed{\text{True}}$: we have $([T]_\gamma^\beta)^{-1}[\mathbf{v}]_\beta = [T^{-1}]_\beta^\gamma[\mathbf{v}]_\beta = [T^{-1}\mathbf{v}]_\gamma$ by the composition formula.

   (n) If $T : V \to V$ and $[T]_\beta^\gamma$ is the identity matrix, then T must be the identity transformation.

   - $\boxed{\text{False}}$: for example, if T is the doubling map and γ is obtained by doubling the vectors in β, then $[T]_\beta^\gamma$ is the identity matrix but T is not the identity map.

   (o) If $T : V \to V$ and $[T]_\beta^\gamma$ is the zero matrix, then T must be the zero transformation.

   - $\boxed{\text{True}}$: if $[T]_\beta^\gamma$ is the zero matrix then $T(\beta_i) = \mathbf{0}$ for every vector $\beta_i \in \beta$. Then since T is zero on a basis, it is zero on all of V.

2. For each linear transformation T and given bases β and γ, find $[T]_\beta^\gamma$:

(a) $T : \mathbb{C}^2 \to \mathbb{C}^3$ given by $T(a,b) = \langle a-b, b-2a, 3b\rangle$, with $\beta = \{\langle 1,0\rangle, \langle 0,1\rangle\}$, $\gamma = \{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$.

- We have $T(1,0) = \langle 1,-2,0\rangle$ and $T(0,1) = \langle -1,1,3\rangle$ so the matrix is $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 0 & 3 \end{bmatrix}}$.

(b) The trace map from $M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ with $\beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right\}$ and $\gamma = \{1\}$.

- We have $T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = 1$, $T(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}) = 0$, $T(\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}) = 0$, and $T(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 5$.
- So the matrix is $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 1 & 0 & 0 & 5 \end{bmatrix}}$.

(c) $T : \mathbb{Q}^4 \to P_4(\mathbb{Q})$ given by $T(a,b,c,d) = a + (a+b)x + (a+3c)x^2 + (2a+d)x^3 + (b+5c+d)x^4$, with β the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.

- We have $T(1,0,0,0) = 1 + x + x^2 + 2x^3$, $T(0,1,0,0) = x + x^4$, $T(0,0,1,0) = 3x^2 + 5x^4$, and $T(0,0,0,1) = x^3 + x^4$. Thus, $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$.

(d) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

- We have $T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, and $T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$. Thus, $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}}$.

(e) The matrix $[T]_\beta^\gamma$ associated to the linear transformation $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ with $P(p) = x^2 p'(x)$, where $\beta = \{1-x, 1-x^2, 1-x^3, x^2+x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$.

- Since $T(1-x) = -x^2$, $T(1-x^2) = -2x^3$, $T(1-x^3) = -3x^4$, $T(x^2+x^3) = 2x^3 + 3x^4$, the matrix is $\boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & 3 \end{bmatrix}}$.

(f) The projection map (see problem 8 of homework 4) on $\mathbb{R}^3$ that maps the vectors $\langle 1,2,1\rangle$ and $\langle 0,-3,1\rangle$ to themselves and sends $\langle 1,1,1\rangle$ to the zero vector, with $\beta = \gamma = \{\langle 1,2,1\rangle, \langle 0,-3,1\rangle, \langle 1,1,1\rangle\}$.

- We have $T(1,2,1) = \langle 1,2,1\rangle$, $T(0,-3,1) = \langle 0,-3,1\rangle$, and $T(1,1,1) = \langle 0,0,0\rangle$.
- So the matrix is simply $[T]_\beta^\gamma = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$.

(g) The same map as in part (f), but relative to the standard basis for $\mathbb{R}^3$.

- One approach is to compute the action of T on the standard basis directly. Another approach is to use the change-of-basis formula: if α is the standard basis and β is the basis from (f), then $Q = [I]_\beta^\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so $[I]_\alpha^\beta = Q^{-1} = \begin{bmatrix} -4 & 1 & 3 \\ -1 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}$. Then $[T]_\alpha^\alpha = Q[T]_\beta^\beta Q^{-1} = \boxed{\begin{bmatrix} -4 & 1 & 3 \\ -5 & 2 & 3 \\ -5 & 1 & 4 \end{bmatrix}}$.

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3. Let $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ be given by $T(p) = x^2 p''(x)$.

   (a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\gamma$.

   - We have $T(1) = 0$, $T(x) = 0$, $T(x^2) = 2x^2$, and $T(x^3) = 6x^3$, so $[T]_\alpha^\gamma = \boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$.

   (b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_\alpha$ and $[T(q)]_\gamma$ and verify that $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

   - We have $[q]_\alpha = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and $T(q) = -2x^2 + 12x^3$, so $[T(q)]_\gamma = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 12 \\ 0 \end{bmatrix}$. Indeed, $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

   Notice that $T = SU$ where $U : P_3(\mathbb{R}) \to P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \to P_4(\mathbb{R})$ has $S(p) = x^2 p(x)$.

   (c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_\beta^\gamma$, and $[U]_\alpha^\beta$ and then verify that $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$.

   - Since $S(1) = x^2$ and $S(x) = x^3$ we have $[S]_\beta^\gamma = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$, and likewise since $U(1) = 0$, $U(x) = 0$, $U(x^2) = 2$, and $U(x^3) = 6x$, we see $[U]_\alpha^\beta = \boxed{\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}}$. Then $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$ as claimed.

   (d) Which of S, T, and U are onto? One-to-one? Isomorphisms?

   - The map U is onto (since its image is all of $P_1(\mathbb{R})$), but S and T are not onto.
   - Also, S is one-to-one, since its kernel is trivial, but U and T both have nonzero elements in their kernels, so they are not one-to-one.
   - Since none of the maps is both one-to-one and onto, none of them are isomorphisms.

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4. Let F be a field and $n \geq 2$ be an integer. Recall that we say two matrices A and B are similar if there exists an invertible matrix Q with $B = Q^{-1}AQ$.

   (a) Show that if A and B are similar matrices in $M_{n\times n}(F)$, then $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. [Hint: You may use the fact that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.]

   - If $B = Q^{-1}AQ$, then $\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = \dfrac{1}{\det(Q)}\det(A)\det(Q) = \det(A)$.
   - Likewise, using the property in the hint with $C = Q^{-1}$ and $D = AQ$, we see $\mathrm{tr}(B) = \mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(AQQ^{-1}) = \mathrm{tr}(A)$.

   (b) Show that "being similar" is an equivalence relation on $M_{n\times n}(F)$.

   - Reflexive: Every matrix is similar to itself, since $A = I_n^{-1} A I_n$.
   - Symmetric: If A is similar to B, so that $B = Q^{-1}AQ$, then $A = QBQ^{-1} = (Q^{-1})^{-1}BQ^{-1}$, so B is similar to A.
   - Transitive: If A is similar to B and B is similar to C, say with $A = Q^{-1}BQ$ and $B = R^{-1}CR$, then $A = R^{-1}Q^{-1}CQR = (QR)^{-1}C(QR)$, so A is similar to C.

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5. Let V be a vector space and $T : V \to V$ be linear.

(a) If V is finite-dimensional and $\ker(T) \cap \operatorname{im}(T) = \{\mathbf{0}\}$, prove in fact that $V = \ker(T) \oplus \operatorname{im}(T)$. [Hint: Use problem 4 from homework 3.]

- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\ker(T)$ and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis for $\operatorname{im}(T)$.
- Since $\ker(T) \cap \operatorname{im}(T) = \{\mathbf{0}\}$, by problem 4(a) of homework 3, we see that the union $\beta \cup \gamma$ is a basis for $\ker(T) + \operatorname{im}(T)$.
- But then by the nullity-rank theorem, $\ker(T) + \operatorname{im}(T)$ has dimension $m + n = \dim(\ker T) + \dim(\operatorname{im} T) = \dim(V)$, and so $\ker(T) + \operatorname{im}(T) = V$.
- This means $V = \ker(T) \oplus \operatorname{im}(T)$, as claimed.

(b) Show that the result of (a) is not necessarily true if V is infinite-dimensional.

- There are various possible counterexamples.
- One possibility is the right-shift map $R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$ whose kernel is zero but which is not onto: so $\ker(R) \cap \operatorname{im}(R) = \{\mathbf{0}\}$ but $V \neq \ker(R) + \operatorname{im}(R)$.
- Another is the antiderivative map on polynomials: $A(p) = \int_0^x p(t)\,dt$. Again, the kernel is zero but the map is not onto (since the image contains no nonzero constants).

(c) If V is finite-dimensional and $V = \ker(T) + \operatorname{im}(T)$, prove in fact that $V = \ker(T) \oplus \operatorname{im}(T)$.

- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\ker(T)$ and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a basis for $\operatorname{im}(T)$.
- Since $V = \ker(T)+\operatorname{im}(T)$ has dimension $\dim(V) = \dim(\ker T)+\dim(\operatorname{im} T) = m+n$, since $\beta\cup\gamma$ spans V it must be linearly independent. Then by problem 4(b) of homework 3, we see that $\ker(T)\cap\operatorname{im}(T) = \{\mathbf{0}\}$.
- This means by problem 4(a) of homework 3, we see that the union $\beta\cup\gamma$ is a basis for $\ker(T)+\operatorname{im}(T)$.
- This means $V = \ker(T) \oplus \operatorname{im}(T)$, as claimed.

(d) Show that the result of (c) is not necessarily true if V is infinite-dimensional.

- There are various possible counterexamples.
- One possibility is the left-shift map $L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$ whose kernel is the sequences $(a_1, 0, 0, \dots)$ but which is also onto: so $V = \ker(T) + \operatorname{im}(T)$ but $\ker(T) \cap \operatorname{im}(T) \neq \{\mathbf{0}\}$.
- Another is the derivative map on polynomials: $A(p) = p'$. Again, the kernel is nonzero but the map is onto.

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6. Let F be a field and d be a positive integer.

(a) Show that any polynomial in $P_d(F)$ with more than d distinct roots must be the zero polynomial. [Hint: Use the factor theorem.]

- Suppose $p(x)$ is a polynomial of degree at most d with $d + 1$ distinct roots $r_1, \dots r_{d+1}$.
- By the factor theorem, $p(x)$ is divisible by $x - r_1$, say $p(x) = (x - r_1)p_1(x)$ where p_1 has degree $d - 1$.
- Then since $0 = p(r_2) = (r_2 - r_1)p_1(r_2)$ we must have p_1 divisible by $x - r_2$, so $p_1(x) = (x - r_2)p_2(x)$ where p_2 has degree $d - 2$, meaning $p(x) = (x - r_1)(x - r_2)p_2(x)$.
- Iterating this procedure (or by a trivial induction) shows that $p(x) = (x - r_1)(x - r_2)\cdots(x - r_d)p_d(x)$ for some polynomial $p_d(x)$ of degree 0. But setting $x = r_{d+1}$ shows that $p_d(r_{d+1}) = 0$ so since p_d is constant, it must be zero. Then $p(x) = 0$ is the zero polynomial.

Now let $a_0, a_1, \dots, a_d$ be distinct elements of F and consider the linear transformation $T : P_d(F) \to F^{d+1}$ given by $T(p) = (p(a_0), p(a_1), \dots, p(a_d))$.

(b) Show that $\ker(T) = \{\mathbf{0}\}$ and deduce that T is an isomorphism.

- If p is in $\ker(T)$, then $p(a_0) = p(a_1) = \cdots = p(a_d) = 0$, meaning that p has $d + 1$ roots. Thus, by (a), p must be the zero polynomial. Thus $\ker(T) = \{\mathbf{0}\}$ as desired.
- Then by the nullity-rank theorem, we have $\dim(\operatorname{im} T) = d + 1 = \dim(F^{d+1})$, so T is also onto, and is therefore an isomorphism.

(c) Conclude that, for any list of $d+1$ points $(a_0, b_0), \dots, (a_d, b_d)$ with distinct first coordinates, there exists a unique polynomial of degree at most d having the property that $p(a_i) = b_i$ for each $0 \le i \le d$.

- This is (ultimately) just a restatement of the fact that T is an isomorphism from (b): since T is onto, there is a polynomial $p(x)$ in $P_d(F)$ with $T(p) = (b_0, b_1, \ldots, b_d)$, which is the same as saying that $p(a_0) = b_0, \ldots, p(a_d) = b_d$.
- Furthermore, since T is one-to-one, there is only one such polynomial.

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7. [Challenge] The goal of this problem is to discuss dual vector spaces. If V is an F-vector space, its dual space V^* is the set of F-valued linear transformations $T : V \to F$. Observe that V^* is a vector space under pointwise addition and scalar multiplication.

   If $\beta = \{\mathbf{e}_i\}_i$ is a basis of V, its associated dual set is the set $\beta^* = \{e_i^*\}_i$ where $e_i^* : V \to F$ is defined by $e_i^*(\mathbf{e}_i) = 1$ and $e_i^*(\mathbf{e}_j) = 0$ for $i \neq j$. (In other words, e_i^* is the linear transformation that sends $\mathbf{e}_i$ to 1 and all of the other basis vectors in β to 0.)

   (a) Show that the dual set β^* is linearly independent.

   - Suppose we have a linear dependence $a_1 e_i^* + a_2 e_2^* + \cdots + a_n e_n^* = 0$, meaning that this function $a_1 e_1^* + a_2 e_2^* + \cdots + a_n e_n^*$ evaluates to zero on every vector in V.
   - In particular, evaluating this function on the vector $\mathbf{e}_i$ yields the coefficient a_i.

   (b) If V is finite-dimensional, let $f \in V^*$. Show that $f = \sum_i f(\mathbf{e}_i) e_i^*$. Deduce that the dual set β^* is a basis of V^* and that $\dim(V^*) = \dim(V)$. [Hint: Show f agrees with the sum on each $\mathbf{e}_j$.]

   - Let $g = \sum_i f(\mathbf{e}_i) e_i^*$. Then $g(\mathbf{e}_j) = [\sum_i f(\mathbf{e}_i) e_i^*](\mathbf{e}_j) = \sum_i f(\mathbf{e}_i) \cdot e_i^*(\mathbf{e}_j)$, but $e_i^*(\mathbf{e}_j) = 0$ except when $i = j$. So the sum reduces just to the term where $i = j$: namely, $g(\mathbf{e}_j) = f(\mathbf{e}_j) \cdot e_j^*(\mathbf{e}_j) = f(\mathbf{e}_j)$.
   - Therefore, we see that g and f agree on each basis vector $\mathbf{e}_j$, so since a linear transformation is characterized by its values on a basis, g and f are equal as functions.
   - The second part follows immediately, since it implies that the $\{e_i^*\}_i$ span V^*, so by part (a), they are a basis for V^*. For the last part we simply observe that there are the same number of e_i^* as $\mathbf{e}_i$, so $\dim(V^*) = \dim(V)$.

   Part (b) shows that when V is finite-dimensional, the association $\{\mathbf{e}_i\}_i \to \{e_i^*\}_i$ extends to an isomorphism of V with V^*. However, this isomorphism depends on a choice of a specific basis for V. Iterating this map shows that V is also isomorphic with its double-dual V^{**}: interestingly, however, there exists a natural isomorphism of V with V^{**} that does not require a specific choice of basis.

   (c) For $\mathbf{v} \in V$, define the "evaluation-at-$\mathbf{v}$ map" $\hat{\mathbf{v}} : V^* \to F$ by setting $\hat{\mathbf{v}}(f) = f(\mathbf{v})$ for every $f \in V^*$: then $\hat{\mathbf{v}}$ is an element of V^{**}. When V is finite-dimensional, show that the map $\varphi : V \to V^{**}$ with $\varphi(\mathbf{v}) = \hat{\mathbf{v}}$ is an isomorphism. [Hint: Show φ is linear and one-to-one.]

   - First, φ is linear: for $\mathbf{v}, \mathbf{w} \in V$ and $f \in V^*$ we have $\varphi(\mathbf{v} + \alpha\mathbf{w})(f) = f(\mathbf{v} + \alpha\mathbf{w}) = f(\mathbf{v}) + \alpha f(\mathbf{w}) = \varphi(\mathbf{v})(f) + \alpha\varphi(\mathbf{w})(f)$.
   - Second, φ is one-to-one: suppose $\mathbf{v} \in V$ is nonzero. Then we may extend it to a basis β and take $f : V \to F$ to be the linear transformation mapping $\mathbf{v}$ to 1 and the rest of β to 0: this means $f(\mathbf{v}) = 1$ and thus $\hat{\mathbf{v}}(f) = 1$, meaning that $\varphi(\hat{\mathbf{v}})(f) = 1$, so $\varphi(\hat{\mathbf{v}})$ is not zero.
   - Finally, since $\dim(V) = \dim(V^{**})$, since φ is one-to-one and linear, it is an isomorphism.

   Essentially all of the results of (b) and (c) fail when V is infinite-dimensional.

   (d) For $V = F[x]$ with basis $\beta = \{1, x, x^2, x^3, \ldots\}$, show that the linear transformation T with $T(p) = p(1)$ is not in $\mathrm{span}(\beta^*)$. Deduce that β^* is not a basis of V^*.

   - Note that the element e_i of the dual set β^* evaluates to 1 on x^i and 0 on other powers of x.
   - The linear transformation T, on the other hand, evaluates to 1 on all powers of x. It therefore cannot be written as a finite linear combination of the elements e_i^*, since any such element $a_0 e_0^* + \cdots + a_n e_n^*$ evaluates to zero on x^{n+1}.
   - Therefore, T is not in $\mathrm{span}(\beta^*)$, so β^* does not span V^* hence is not a basis.

   Remark: In part (d), it can in fact be shown that the dimension of V^* is uncountable, while the dimension of V is countable, so V^* and V are not even isomorphic.
