E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 5, due Fri Feb 14th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that the vector spaces U, V, W are finite-dimensional over the field F, the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that S, T are linear transformations. Identify each of the following statements as true or false:

   (a) The space $\mathcal{L}(V, W)$ of all linear transformations from V to W has dimension $\dim V \cdot \dim W$.

   (b) If A is an $m \times n$ matrix of rank r, then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension r.

   (c) If A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $\text{rank}(A) < n$.

   (d) If A is an $n \times n$ matrix of rank n, then the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.

   (e) If the columns of A are all scalar multiples of some vector $\mathbf{v}$, then $\text{rank}(A) \leq 1$.

   (f) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ is an element of $M_{m \times n}(F)$.

   (g) If $[S]_\alpha^\beta = [T]_\alpha^\beta$ then $S = T$.

   (h) If $[T]_\alpha^\beta = [T]_\gamma^\delta$ then $\alpha = \gamma$ and $\beta = \delta$.

   (i) If $S : V \to W$ and $T : V \to W$ then $[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$.

   (j) If $T : V \to W$ and $\mathbf{v} \in V$, then $[T]_\alpha^\beta [\mathbf{v}]_\beta = [T\mathbf{v}]_\alpha$.

   (k) If $S : V \to W$ and $T : U \to V$, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$.

   (l) If $T : V \to V$ has an inverse T^{-1}, then $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

   (m) If $T : V \to V$ has an inverse T^{-1}, then for any $\mathbf{v} \in V$, $[T^{-1}\mathbf{v}]_\gamma = ([T]_\gamma^\beta)^{-1}[\mathbf{v}]_\beta$.

   (n) If $T : V \to V$ and $[T]_\beta^\gamma$ is the identity matrix, then T must be the identity transformation.

   (o) If $T : V \to V$ and $[T]_\beta^\gamma$ is the zero matrix, then T must be the zero transformation.

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2. For each linear transformation T and given bases β and γ, find $[T]_\beta^\gamma$:

   (a) $T : \mathbb{C}^2 \to \mathbb{C}^3$ given by $T(a, b) = \langle a - b, b - 2a, 3b \rangle$, with $\beta = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$, $\gamma = \{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$.

   (b) The trace map from $M_{2 \times 2}(\mathbb{R}) \to \mathbb{R}$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ and $\gamma = \{1\}$.

   (c) $T : \mathbb{Q}^4 \to P_4(\mathbb{Q})$ given by $T(a, b, c, d) = a + (a + b)x + (a + 3c)x^2 + (2a + d)x^3 + (b + 5c + d)x^4$, with β the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.

   (d) $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   (e) The matrix $[T]_\beta^\gamma$ associated to the linear transformation $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ with $P(p) = x^2 p'(x)$, where $\beta = \{1 - x, 1 - x^2, 1 - x^3, x^2 + x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$.

   (f) The projection map (see problem 8 of homework 4) on $\mathbb{R}^3$ that maps the vectors $\langle 1, 2, 1 \rangle$ and $\langle 0, -3, 1 \rangle$ to themselves and sends $\langle 1, 1, 1 \rangle$ to the zero vector, with $\beta = \gamma = \{\langle 1, 2, 1 \rangle, \langle 0, -3, 1 \rangle, \langle 1, 1, 1 \rangle\}$.

   (g) The same map as in part (f), but relative to the standard basis for $\mathbb{R}^3$.

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3. Let $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ be given by $T(p) = x^2 p''(x)$.

   (a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\gamma$.

   (b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_\alpha$ and $[T(q)]_\gamma$ and verify that $[T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha$.

   Notice that $T = SU$ where $U : P_3(\mathbb{R}) \to P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \to P_4(\mathbb{R})$ has $S(p) = x^2 p(x)$.

   (c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_\beta^\gamma$, and $[U]_\alpha^\beta$ and then verify that $[T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\alpha^\beta$.

   (d) Which of S, T, and U are onto? One-to-one? Isomorphisms?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let F be a field and $n \geq 2$ be an integer. Recall that we say two matrices A and B are similar if there exists an invertible matrix Q with $B = Q^{-1}AQ$.

   (a) Show that if A and B are similar matrices in $M_{n\times n}(F)$, then $\det(A) = \det(B)$ and $\text{tr}(A) = \text{tr}(B)$. [Hint: You may use the fact that $\text{tr}(CD) = \text{tr}(DC)$.]

   (b) Show that "being similar" is an equivalence relation on $M_{n\times n}(F)$.

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5. Let V be a vector space and $T : V \to V$ be linear.

   (a) If V is finite-dimensional and $\ker(T) \cap \text{im}(T) = \{\mathbf{0}\}$, prove in fact that $V = \ker(T) \oplus \text{im}(T)$. [Hint: Use problem 4 from homework 3.]

   (b) Show that the result of (a) is not necessarily true if V is infinite-dimensional.

   (c) If V is finite-dimensional and $V = \ker(T) + \text{im}(T)$, prove in fact that $V = \ker(T) \oplus \text{im}(T)$.

   (d) Show that the result of (c) is not necessarily true if V is infinite-dimensional.

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6. Let F be a field and d be a positive integer.

   (a) Show that any polynomial in $P_d(F)$ with more than d distinct roots must be the zero polynomial. [Hint: Use the factor theorem.]

   Now let $a_0, a_1, \dots, a_d$ be distinct elements of F and consider the linear transformation $T : P_d(F) \to F^{d+1}$ given by $T(p) = (p(a_0), p(a_1), \dots, p(a_d))$.

   (b) Show that $\ker(T) = \{\mathbf{0}\}$ and deduce that T is an isomorphism.

   (c) Conclude that, for any list of $d+1$ points $(a_0, b_0), \dots, (a_d, b_d)$ with distinct first coordinates, there exists a unique polynomial of degree at most d having the property that $p(a_i) = b_i$ for each $0 \leq i \leq d$.

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7. [Challenge] The goal of this problem is to discuss dual vector spaces. If V is an F-vector space, its <u>dual space</u> V^* is the set of F-valued linear transformations $T : V \to F$. Observe that V^* is a vector space under pointwise addition and scalar multiplication.

   If $\beta = \{\mathbf{e}_i\}_i$ is a basis of V, its associated <u>dual set</u> is the set $\beta^* = \{e_i^*\}_i$ where $e_i^* : V \to F$ is defined by $e_i^*(\mathbf{e}_i) = 1$ and $e_i^*(\mathbf{e}_j) = 0$ for $i \neq j$. (In other words, e_i^* is the linear transformation that sends $\mathbf{e}_i$ to 1 and all of the other basis vectors in β to 0.)

   (a) Show that the dual set β^* is linearly independent.

   (b) If V is finite-dimensional, let $f \in V^*$. Show that $f = \sum_i f(\mathbf{e}_i)e_i^*$. Deduce that the dual set β^* is a basis of V^* and that $\dim(V^*) = \dim(V)$. [Hint: Show f agrees with the sum on each $\mathbf{e}_j$.]

   Part (b) shows that when V is finite-dimensional, the association $\{\mathbf{e}_i\}_i \to \{e_i^*\}_i$ extends to an isomorphism of V with V^*. However, this isomorphism depends on a choice of a specific basis for V. Iterating this map shows that V is also isomorphic with its double-dual V^{**}: interestingly, however, there exists a natural isomorphism of V with V^{**} that does not require a specific choice of basis.

   (c) For $\mathbf{v} \in V$, define the "evaluation-at-$\mathbf{v}$ map" $\hat{\mathbf{v}} : V^* \to F$ by setting $\hat{\mathbf{v}}(f) = f(\mathbf{v})$ for every $f \in V^*$: then $\hat{\mathbf{v}}$ is an element of V^{**}. When V is finite-dimensional, show that the map $\varphi : V \to V^{**}$ with $\varphi(\mathbf{v}) = \hat{\mathbf{v}}$ is an isomorphism. [Hint: Show φ is linear and one-to-one.]

   Essentially all of the results of (b) and (c) fail when V is infinite-dimensional.

   (d) For $V = F[x]$ with basis $\beta = \{1, x, x^2, x^3, \dots\}$, show that the linear transformation T with $T(p) = p(1)$ is not in $\text{span}(\beta^*)$. Deduce that β^* is not a basis of V^*.

   Remark: In part (d), it can in fact be shown that the dimension of V^* is uncountable, while the dimension of V is countable, so V^* and V are not even isomorphic.
