1. Assume that V and W are arbitrary vector spaces, not necessarily finite-dimensional, over the scalar field F. Identify each of the following statements as true or false:

   (a) If $T : V \to W$ is a linear transformation, then $T(\mathbf{0}_V) = \mathbf{0}_W$.
   - $\boxed{\text{True}}$: this is a property we proved about all linear transformations.

   (b) If $T : V \to W$ has $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$ for every $\mathbf{a}, \mathbf{b} \in V$ then T is a linear transformation.
   - $\boxed{\text{False}}$: a linear transformation must also respect scalar multiplication. An explicit counterexample is given by the complex conjugation map $T : \mathbb{C} \to \mathbb{C}$ with $T(a + bi) = a - bi$. This map respects addition but not scalar multiplication, since $T(i \cdot 1) = -i \neq i = i \cdot T(1)$.

   (c) If $T : V \to W$ has $T(r\mathbf{a}) = rT(\mathbf{a})$ for every $r \in F$ and every $\mathbf{a} \in V$ then T is a linear transformation.
   - $\boxed{\text{False}}$: a linear transformation must also respect addition of vectors. An explicit counterexample is given by the map $T : \mathbb{C} \to \mathbb{C}$ (with $F = \mathbb{R}$) with $T(a + 0i) = 0$ and $T(a + bi) = a + bi$ for $b \neq 0$. This map respects scaling by real constants, but not addition.

   (d) If T is a linear transformation such that $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, then T is one-to-one.
   - $\boxed{\text{True}}$: this is a result we established in class. In fact, it is an if-and-only if.

   (e) If $T : V \to V$ is a one-to-one linear transformation, then $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
   - $\boxed{\text{True}}$: this is the converse of (d), and is also a result we established in class.

   (f) If $T : V \to W$ is linear and $\dim(\text{im}(T)) = \dim(W)$, then T is onto.
   - $\boxed{\text{False}}$: this is only true for finite-dimensional vector spaces.

   (g) If $T : V \to W$ is linear and S spans V, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ spans W.
   - $\boxed{\text{False}}$: we showed $T(S)$ is a spanning set for $\text{im}(T)$, but if T is not onto then $T(S)$ cannot span W.

   (h) If $T : V \to W$ is linear and S is linearly independent in V, then $T(S)$ is a linearly independent in W.
   - $\boxed{\text{False}}$: for example, if T is the zero map then S will never be linearly independent.

   (i) If $T : V \to W$ is linear and S is a basis for V, then $T(S)$ is a basis for W.
   - $\boxed{\text{False}}$: from (g) and (h), $T(S)$ need not be linearly independent nor span W.

   (j) For any $\mathbf{v}_1, \mathbf{v}_2 \in V$ and any $\mathbf{w}_1, \mathbf{w}_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.
   - $\boxed{\text{False}}$: if $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent, then the same dependence will hold between $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$, so the values cannot be chosen arbitrarily.

   (k) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 2 and whose rank is 2.
   - $\boxed{\text{False}}$: by the nullity-rank theorem, the nullity plus the rank must be 5, but $2 + 2 = 4$.

   (l) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 4 and whose rank is 1.
   - $\boxed{\text{True}}$: one such map is $T(a, b, c, d, e) = (a, 0, 0, 0)$.

   (m) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 1 and whose rank is 4.
   - $\boxed{\text{False}}$: although the nullity-rank theorem does not pose any issues, the rank cannot be 4 because the target space $\mathbb{R}^3$ only has dimension 3.

   (n) If $T : V \to W$ is linear and for any $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.
   - $\boxed{\text{True}}$: this statement is equivalent to saying that T has an inverse function $T^{-1} : W \to V$, which is the same as saying that T is an isomorphism.

   (o) If V is isomorphic to W, then $\dim(V) = \dim(W)$.
   - $\boxed{\text{True}}$: an isomorphism maps a basis of V to a basis of W, so $\dim(V) = \dim(W)$.

2. Calculate the following:

(a) If $S = \{\langle 2,1,-1\rangle, \langle -1,2,3\rangle, \langle -2,3,5\rangle, \langle 4,1,-3\rangle\}$, find a subset of S that is a basis for span(S) in $\mathbb{R}^3$.

- We simply row-reduce the matrix whose columns are the vectors in S:

$$\begin{bmatrix} 2 & -1 & -2 & 4 \\ 1 & 2 & 3 & 1 \\ -1 & 3 & 5 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & 8/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since the first and second columns are pivotal, we conclude that the vectors $\boxed{\langle 2,1,-1\rangle, \langle -1,2,3\rangle}$ are a basis for the column space, which is the same as span(S).

(b) Extend the set $S = \{\langle 1,2,1,1\rangle, \langle -1,2,2,2\rangle, \langle -2,1,2,2\rangle\}$ to a basis of $\mathbb{Q}^4$.

- We extend S to a spanning set, and then reduce the result to a basis, by appending the standard basis to S.
- To do this, row-reduce the matrix whose columns are the vectors in S followed by the standard basis:

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & 0 & 3 \\ 0 & 1 & 0 & -3 & 4 & 0 & -5 \\ 0 & 0 & 1 & 2 & -3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

- Since columns 1, 2, 3, and 6 are pivotal, we conclude that we may append the vector corresponding to the 6th column of the original matrix, which is $\langle 0,0,1,0\rangle$.
- Hence we obtain a basis $\boxed{\langle 1,2,1,1\rangle, \langle -1,2,2,2\rangle, \langle -2,1,2,2\rangle, \langle 0,0,1,0\rangle}$.

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3. For each map $T : V \to W$, determine whether or not T is a linear transformation from V to W, and if it is not, identify at least one property that fails:

(a) $V = W = \mathbb{R}^4$, $T(a,b,c,d) = (a-b, b-c, c-d, d-a)$.

- This map $\boxed{\text{is linear}}$ because it is equivalent to left-multiplication by a 4×4 matrix.

(b) $V = W = \mathbb{R}^2$, $T(a,b) = (a, b^2)$.

- This map $\boxed{\text{is not linear}}$ because it does not respect addition or scalar multiplication.

(c) $V = W = M_{2\times 2}(\mathbb{Q})$, $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} A - A \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

- This map $\boxed{\text{is linear}}$: writing $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$, we see that $T(A_1 + cA_2) = B(A_1 + cA_2) - (A_1 + cA_2)C = (BA_1 - A_1C) + c(BA_2 - A_2C) = T(A_1) + cT(A_2)$.

(d) $V = W = \mathbb{C}[x]$, $T(p(x)) = p(x^2) - xp'(x)$.

- This map $\boxed{\text{is linear}}$: one may check it directly, or observe that the maps $T_1(p) = p(x^2)$ and $T_2(p) = xp'(x)$ are both linear separately, hence their difference is also.

(e) $V = W = M_{4\times 4}(\mathbb{F}_2)$, $T(A) = Q^{-1}AQ$, for a fixed 4×4 matrix Q.

- This map $\boxed{\text{is linear}}$: $T(A_1 + cA_2) = Q^{-1}(A_1 + cA_2)Q = Q^{-1}A_1Q + cQ^{-1}A_2Q = T(A_1) + cT(A_2)$.

(f) $V = W = M_{4\times 4}(\mathbb{R})$, $T(A) = A^{-1}QA$, for a fixed 4×4 matrix Q.

- This map $\boxed{\text{is not linear}}$, and is not even defined unless A is invertible. It fails both [T1] and [T2].

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4. For each map $T : V \to W$, (i) show that T is a linear transformation, (ii) find bases for the kernel and image of T, (iii) compute the nullity and rank of T and verify the conclusion of the nullity-rank theorem, and (iv) identify whether T is one-to-one, onto, or an isomorphism.

(a) $T : \mathbb{Q}^2 \to \mathbb{Q}^3$ defined by $T(a, b) = \langle a + b, 2a + 2b, a + b\rangle$.

- This map is equivalent to left-multiplication by a matrix so it is a linear transformation.
- The kernel is the set of vectors $\langle a, b\rangle$ with $T(a, b) = \langle 0, 0, 0\rangle$, so we obtain the system $a + b = 0$, $2a + 2b = 0$, $a + b = 0$ which clearly has the solution $a = -b$. Hence $\ker(T)$ has basis $\boxed{\{\langle 1, -1\rangle\}}$.
- The image is spanned by $\{T(1, 0), T(0, 1)\} = \{\langle 1, 2, 1\rangle, \langle 1, 2, 1\rangle\}$ so we have an obvious basis $\boxed{\{\langle 1, 2, 1\rangle\}}$.
- The nullity is $\boxed{1}$ and the rank is $\boxed{1}$, and indeed $1 + 1 = 2 = \dim_{\mathbb{Q}} \mathbb{Q}^2$.
- Since $\ker(T)$ is not zero T is $\boxed{\text{not one-to-one}}$, and since $\operatorname{im}(T)$ only has dimension 1, T is $\boxed{\text{not onto}}$.

(b) $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.

- This map is left-multiplication by a matrix so it is a linear transformation.
- Since $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$ we see $\ker(T) = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$ with basis $\boxed{\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\}}$.
- Likewise, $\operatorname{im}(T) = \{\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}\}$ which has a natural basis $\boxed{\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}}$.
- The nullity is $\boxed{2}$ and the rank is $\boxed{2}$, and indeed $2 + 2 = 4 = \dim_{\mathbb{R}}(M_{2\times2}(\mathbb{R}))$.
- Since $\ker(T)$ is not zero T is $\boxed{\text{not one-to-one}}$, and since $\operatorname{im}(T)$ only has dimension 2, T is $\boxed{\text{not onto}}$.

(c) $T : P_2(\mathbb{C}) \to P_3(\mathbb{C})$ defined by $T(p) = xp(x) + p'(x)$.

- We have $T(f + cg) = x(f + cg) + (f + cg)' = (xf + f') + c(xg + g') = T(f) + cT(g)$, so T is linear.
- The kernel is the set of polynomials $a + bx + cx^2$ with $b = 0$, $a + 2c = 0$, $b = 0$, $c = 0$, which clearly requires $a = b = c = 0$. Thus, $\ker(T) = \{0\}$, so the empty set $\boxed{\emptyset}$ is a basis.
- The image is spanned by $\{T(1), T(x), T(x^2)\} = \boxed{\{x, x^2 + 1, x^3 + 2x\}}$ which is in fact a basis.
- The nullity is $\boxed{0}$ and the rank is $\boxed{3}$, and indeed $0 + 3 = 3 = \dim_{\mathbb{C}}(P_2(\mathbb{C}))$.
- Since $\ker(T) = 0$, T is $\boxed{\text{one-to-one}}$, but since $\operatorname{im}(T)$ only has dimension 3, T is $\boxed{\text{not onto}}$.

(d) $T : P_3(\mathbb{F}_3) \to P_4(\mathbb{F}_3)$ defined by $T(p) = x^3p''(x)$. [Warning: Note that $3 = 0$ in $\mathbb{F}_3$.]

- We have $T(f + cg) = x^2(f + cg)'' = x^2f'' + c(x^2g'') = T(f) + cT(g)$ so T is linear.
- More explicitly, if $p = a_0 + a_1x + a_2x^2 + a_3x^3$ then $T(p) = x^3(2a_2 + 6a_3) = 2a_2x^3$.
- The kernel is the set of polynomials $p = a_0 + a_1x + a_2x^2 + a_3x^3$ with $T(p) = 0$, which requires $2a_2 = 0$ hence $a_2 = 0$. Thus, $\ker(T)$ has a basis $\boxed{\{1, x, x^3\}}$.
- The image is clearly the set of multiples of x^3 hence has a basis $\boxed{\{x^3\}}$.
- The nullity is $\boxed{3}$ and the rank is $\boxed{1}$, and indeed $3 + 1 = 4 = \dim_{\mathbb{F}_3}(P_3(\mathbb{F}_3))$.
- Since $\ker(T) \neq 0$, T is $\boxed{\text{not one-to-one}}$, and since $\operatorname{im}(T)$ only has dimension 1, T is $\boxed{\text{not onto}}$.

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5. Suppose that $T : V \to W$ is a linear transformation.

(a) If T is onto, show that $\dim(W) \leq \dim(V)$.

- If T is onto, then $\operatorname{im} T = W$, so $\dim(\ker T) + \dim(W) = \dim(V)$. Since $\dim(\ker T) \geq 0$, this means $\dim(W) \leq \dim(V)$.

(b) If T is one-to-one, show that T is an isomorphism from V to $\operatorname{im}(T)$, and deduce that $\dim(V) \leq \dim(W)$.

- For the first part, if T is one-to-one, then $T : V \to \operatorname{im}(T)$ is a one-to-one map that is onto (by definition of $\operatorname{im} T$), meaning it is an isomorphism.
- For the second part, since T is an isomorphism from V to $\operatorname{im}(T)$, we have $\dim(\operatorname{im} T) = \dim(V)$. But since $\operatorname{im}(T)$ is a subspace of W, we see that $\dim(\operatorname{im} T) \leq \dim(W)$, so $\dim(V) = \dim(\operatorname{im} T) \leq \dim(W)$.

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6. Suppose $\dim(V) = n$ and that $T : V \to V$ is a linear transformation with $T^2 = 0$: in other words, that $T(T(\mathbf{v})) = \mathbf{0}$ for every vector $\mathbf{v} \in V$.

   (a) Show that $\mathrm{im}(T)$ is a subspace of $\ker(T)$.

   - Suppose $\mathbf{w}$ is in $\mathrm{im}(T)$. This means that there exists $\mathbf{v}$ with $\mathbf{w} = T(\mathbf{v})$.
   - Then $T(\mathbf{w}) = T(T(\mathbf{v})) = \mathbf{0}$, meaning that $\mathbf{w}$ is in $\ker(T)$. Thus, $\mathrm{im}(T) \subseteq \ker(T)$ so it is a subspace.

   (b) Show that $\dim(\mathrm{im}(T)) \le n/2$.

   - By part (a), $\mathrm{im}(T)$ is a subspace of $\ker(T)$, so taking dimensions gives $\dim(\mathrm{im}(T)) \le \dim(\ker(T))$.
   - By the nullity-rank theorem, $\dim(\mathrm{im}(T)) + \dim(\ker(T)) = n$. Thus, $2\dim(\mathrm{im}(T)) \le \dim(\mathrm{im}(T)) + \dim(\ker(T)) = n$, so $\dim(\mathrm{im}(T)) \le n/2$.

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7. Let F be a field and let V be the vector space of infinite sequences $\{a_n\}_{n\ge 1} = (a_1, a_2, a_3, a_4, \dots)$ of elements of F. Define the <u>left-shift operator</u> $L : V \to V$ via $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, a_5, \dots)$ and the <u>right-shift operator</u> $R : V \to V$ via $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Show that L is a linear transformation that is onto but not one-to-one.

   - We have $L(a_1+cb_1, a_2+cb_2, a_3+cb_3, a_4+cb_4, \dots) = (a_2+cb_2, a_3+cb_3, a_4+cb_4 \dots) = L(a_1, a_2, a_3, \dots) + cL(b_1, b_2, b_3, \dots)$ so L is linear.
   - Also, $L(0, a_1, a_2, a_3, \dots) = (a_1, a_2, a_3, \dots)$ so L is onto. But since $\ker(L) = \{c, 0, 0, 0, \dots\}$ is not trivial, L is not one-to-one.

   (b) Show that R is a linear transformation that is one-to-one but not onto.

   - We have $R(a_1+cb_1, a_2+cb_2, a_3+cb_3, a_4+cb_4, \dots) = (0, a_1+cb_1, a_2+cb_2, a_3+cb_3, \dots) = R(a_1, a_2, a_3, \dots) + cR(b_1, b_2, b_3, \dots)$ so R is linear.
   - If $R(a_1, a_2, a_3, \dots) = 0$ then clearly $a_1 = a_2 = a_3 = \cdots = 0$, so R is one-to-one. But $\mathrm{im}(R)$ consists of only the sequences which have first element zero, so R is not onto.

   (c) Deduce that on infinite-dimensional vector spaces, the conditions of being one-to-one, being onto, and being an isomorphism are not in general equivalent.

   - This follows from (a) and (b), since L is one-to-one but not onto, while R is onto but not one-to-one, and neither one is an isomorphism.

   (d) Verify that $L \circ R$ is the identity map on V, but that $R \circ L$ is not the identity map on V.

   - We have $(L \circ R)(a_1, a_2, a_3, a_4, \dots) = L(0, a_1, a_2, a_3, a_4, \dots) = (a_1, a_2, a_3, a_4, \dots)$ so $L \circ R$ is the identity on every sequence hence on V.
   - But $(R \circ L)(a_1, a_2, a_3, a_4, \dots) = R(a_2, a_3, a_4, a_5, \dots) = (0, a_2, a_3, a_4, \dots)$, which is not equal to the original vector whenever $a_1 \ne 0$. So $R \circ L$ is not the identity map on V.

   (e) Deduce that on infinite-dimensional vector spaces, a linear transformation with a left inverse or a right inverse need not have a two-sided inverse.

   - This follows from (d), since L has a right inverse (namely R) and R has a left inverse (namely L), but neither L nor R has a two-sided inverse because they are not isomorphisms as noted in (c).

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8. A linear transformation $T : V \to V$ such that $T^2 = T$ is called a <u>projection map</u>. The goal of this problem is to give some other descriptions of projection maps.

   (a) Suppose that $T : V \to V$ has the property that there exists a subspace W such that $\mathrm{im}(T) = W$ and T is the identity map when restricted to W. Show that T is a projection map (it is called a <u>projection onto the subspace W</u>).

   - Let $\mathbf{v} \in V$. Then $T(\mathbf{v}) \in \mathrm{im}(T)$, and so T acts as the identity on $T(\mathbf{v})$, which is to say, $T(T(\mathbf{v})) = T(\mathbf{v})$. Since this holds for every vector $\mathbf{v} \in V$, this means $T^2 = T$, so T is a projection map.

   (b) Conversely, suppose T is a projection map. Show that T is a projection onto the subspace $W = \mathrm{im}(T)$.

   - By definition we have $\mathrm{im}(T) = W$. Also, for any $\mathbf{w} \in W$ we have $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$.
   - Then since T is a projection map, $T(\mathbf{w}) = T^2(\mathbf{v}) = T(\mathbf{v}) = \mathbf{w}$, so T acts as the identity on W.

(c) Suppose that T is a projection map. Prove that $V = \ker(T) \oplus \operatorname{im}(T)$. [Hint: Write $\mathbf{v} = [\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$.]

- To show that $V = \ker(T) \oplus \operatorname{im}(T)$ we must show $V = \ker(T) + \operatorname{im}(T)$ and $\ker(T) \cap \operatorname{im}(T) = \{\mathbf{0}\}$.
- For the first part, following the hint observe that $[\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$. Then $T(\mathbf{v} - T(\mathbf{v})) = T(\mathbf{v}) - T^2(\mathbf{v}) = \mathbf{0}$, so we see $\mathbf{v} - T(\mathbf{v}) \in \ker(T)$.
- Since clearly $T(\mathbf{v}) \in \operatorname{im}(T)$, we see $\mathbf{v} = [\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$ is the sum of an element of $\ker(T)$ and an element of $\operatorname{im}(T)$, whence $V = \ker(T) + \operatorname{im}(T)$.
- For the second part, suppose $\mathbf{v}$ is in $\ker(T) \cap \operatorname{im}(T)$. Then $T(\mathbf{v}) = \mathbf{0}$ and there exists some $\mathbf{w}$ in V with $T(\mathbf{w}) = \mathbf{v}$. But then $\mathbf{v} = T(\mathbf{w}) = T(T(\mathbf{w})) = T(\mathbf{v}) = \mathbf{0}$.
- Thus, $\ker(T) \cap \operatorname{im}(T) = \{\mathbf{0}\}$, and so $V = \ker(T) \oplus \operatorname{im}(T)$ as claimed.

Remark: Projection maps are so named because they represent the geometric idea of projection. For example, in the event that $W = \operatorname{im}(T)$ is one-dimensional, the corresponding projection map T represents projecting onto that line.

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9. [Challenge] The goal of this problem is to demonstrate some bizarre things one can do with infinite bases.

(a) Show that $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \mathbb{C}$. Deduce that there exists a $\mathbb{Q}$-vector space isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$. [Hint: Use the fact that finite-dimensional $\mathbb{Q}$-vector spaces are countable.]

- Let $\beta = \{\mathbf{v}_j\}_{j \in J}$ be a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Note that J must be infinite because any finite-dimensional vector space over $\mathbb{Q}$ is countable, whereas $\mathbb{R}$ is uncountable.
- We claim that if we define $i\beta = \{i\mathbf{v}_j\}_{j \in J}$ then $\beta \cup i\beta$ is a basis for $\mathbb{C}$ as a $\mathbb{Q}$-vector space.
- To see $\beta \cup i\beta$ spans, if $z = a + bi \in \mathbb{C}$, then we may write $a = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ and $b = b_1\mathbf{w}_1 + \cdots + b_m\mathbf{w}_m$ for some $\mathbf{v}_i, \mathbf{w}_i \in \beta$. But then $z = a + bi = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + b_1 i\mathbf{w}_1 + \cdots + b_m i\mathbf{w}_m$ is in the span of $\beta \cup i\beta$.
- To see $\beta \cup i\beta$ is linearly independent, spans, if $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + b_1 i\mathbf{w}_1 + \cdots + b_m i\mathbf{w}_m = 0$, then the real and imaginary parts must both be zero. But independence of β and $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{w}_1 + \cdots + b_m\mathbf{w}_m = 0$ implies $a_1 = \cdots = a_n = b_1 = \cdots = b_m = 0$.
- So $\beta \cup i\beta$ is a basis for $\mathbb{C}$ as a $\mathbb{Q}$-vector space. This means $\dim_{\mathbb{Q}} \mathbb{C} = 2\dim_{\mathbb{Q}} \mathbb{R}$, but since the latter dimension is infinite, it also equals $\dim_{\mathbb{Q}} \mathbb{R}$ by standard properties of infinite sets.
- The existence of the vector space isomorphism then follows immediately, since spaces of equal dimension are isomorphic.

We will now use this isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$ to define a different vector space structure on $\mathbb{C}$. Intuitively, the idea is to start with the set $\mathbb{R}$ as a vector space over itself, and then use the isomorphism φ^{-1} to relabel the vectors as complex numbers, but keep the scalars as real numbers.

(b) Let V be the set of complex numbers with the addition operation $z_1 \oplus z_2 = z_1 + z_2$ and scalar multiplication defined as follows: for $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$, set $\alpha \odot z = \varphi^{-1}[\alpha\varphi(z)]$. Show $(V, \oplus, \odot)$ is an $\mathbb{R}$-vector space.

- The axioms [V1]-[V4] only concern addition so they follow trivially from properties of complex number addition. Note also that φ and φ^{-1} are both additive, which is actually the only property that we will need.
- [V5]: We have $\alpha \odot (\beta \odot z) = \alpha \odot \varphi^{-1}[\beta\varphi(z)] = \varphi^{-1}[\alpha\varphi[\varphi^{-1}[\beta\varphi(z)]]]] = \varphi^{-1}[\alpha\beta\varphi(z)]] = (\alpha\beta) \odot z$.
- [V6]: We have $(\alpha + \beta) \odot z = \varphi^{-1}[(\alpha + \beta)\varphi(z)] = \varphi^{-1}[\alpha\varphi(z)] + \varphi^{-1}[\beta\varphi(z)] = \alpha \odot z + \beta \odot z$.
- [V7]: We have $\alpha \odot (z + w) = \varphi^{-1}[\alpha\varphi(z + w)] = \varphi^{-1}[\alpha\varphi(z) + \alpha\varphi(w)] = \varphi^{-1}[\alpha\varphi(z)] + \varphi^{-1}[\alpha\varphi(w)] = \alpha \odot z + \alpha \odot w$.
- [V8]: We have $1 \odot z = \varphi^{-1}[1\varphi(z)] = \varphi^{-1}[\varphi(z)] = z$.

(c) Using the vector space structure defined in (b), show that $\dim_{\mathbb{R}} V = 1$.

- We show that the set $\{1\}$ is a basis. Clearly it is linearly independent since $1 \neq 0$.
- To see it spans, observe that for any $z \in \mathbb{C}$, we have $\varphi(z) \odot 1 = \varphi^{-1}[\varphi(z)\varphi(1)] = \varphi^{-1}[\varphi(z)] = z$ because $\varphi(1) = 1$. Therefore, every vector $z \in \mathbb{C}$ is a scalar multiple of 1, so $\{1\}$ spans V.

Remark: The point of (c) is that by changing the definition of scalar multiplication, we can make $\mathbb{C}$ into a 1-dimensional $\mathbb{R}$-vector space. By doing a similar thing in the reverse order, we could even make $\mathbb{R}$ into a 2-dimensional $\mathbb{C}$-vector space.
