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1. Identify each of the following statements as true or false:

   (a) Every vector space contains a zero vector.

   - $\boxed{\text{True}}$: this is one of the vector space axioms.

   (b) In any vector space, $\alpha\mathbf{v} = \alpha\mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$.

   - $\boxed{\text{False}}$: for example we have $0 \cdot \mathbf{v} = 0 \cdot \mathbf{w}$ for any $\mathbf{v}, \mathbf{w}$.

   (c) In any vector space, $\alpha\mathbf{v} = \beta\mathbf{v}$ implies that $\alpha = \beta$.

   - $\boxed{\text{False}}$: for example we have $1 \cdot \mathbf{0} = 2 \cdot \mathbf{0}$.

   (d) If U is a subspace of V and V is a subspace of W, then U is a subspace of W.

   - $\boxed{\text{True}}$: this follows from the subspace criterion (or even just the definition of subspace).

   (e) The empty set is a subspace of any vector space.

   - $\boxed{\text{False}}$: subspaces are by definition not empty.

   (f) The intersection of two subspaces is always a subspace.

   - $\boxed{\text{True}}$: the intersection of any collection of subspaces is a subspace.

   (g) The union of two subspaces is always a subspace.

   - $\boxed{\text{False}}$: for example, the union of $\{\langle x, 0\rangle\}$ and $\{\langle 0, y\rangle\}$ in $\mathbb{R}^2$ is not a subspace.

   (h) The union of two subspaces is never a subspace.

   - $\boxed{\text{False}}$: for example if we take two subspaces $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ is still a subspace.

   (i) The span of the empty set is the empty set.

   - $\boxed{\text{False}}$: the span of the empty set is the zero subspace containing only $\mathbf{0}$, which is different from the empty set.

   (j) The span of the zero vector is the zero subspace.

   - $\boxed{\text{True}}$: the only vector spanned by the zero vector is the zero vector itself.

   (k) If S is any subset of V, then $\text{span}(S)$ is the intersection of all subspaces of V containing S.

   - $\boxed{\text{True}}$: this is one of the basic facts about spans.

   (l) If S is any subset of V, then $\text{span}(S)$ always contains the zero vector.

   - $\boxed{\text{True}}$: the span is always a subspace, so it contains the zero vector (even if S is empty).

   (m) Any set containing the zero vector is linearly independent.

   - $\boxed{\text{False}}$: in fact the opposite is true, any set set containing the zero vector is linearly dependent!

   (n) Any subset of a linearly independent set is linearly independent.

   - $\boxed{\text{True}}$: any linear dependence in the subset would give one in the original set.

   (o) Any subset of a linearly dependent set is linearly dependent.

   - $\boxed{\text{False}}$: removing elements from a linearly dependent set could certainly yield a linearly independent set (e.g., the empty set!).

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2. Determine whether or not each given set S is a subspace of the given vector space V. For each set that is not a subspace, identify at least one part of the subspace criterion that fails.

   (a) $V = \mathbb{R}^4$, $S =$ the vectors $\mathbf{v}$ in $\mathbb{R}^4$ with $\mathbf{v} \cdot \langle 1, 0, 1, 1 \rangle = 2$.

      - This set $\boxed{\text{is not a subspace}}$ because it does not contain the zero vector. (It also fails the other two parts of the subspace criterion.)

   (b) $V =$ real-valued functions on $[0, 1]$, $S =$ the functions with $f''(x) = f(x)$.

      - This set $\boxed{\text{is a subspace}}$ because it contains the zero function and is closed under addition and scalar multiplication.

   (c) $V = \mathbb{C}^5$, $S =$ the vectors $\langle a, b, c, d, e \rangle$ with $e = a + b + c$ and $b = c = d$.

      - This set $\boxed{\text{is a subspace}}$ because it contains the zero vector and is closed under addition and scalar multiplication.

   (d) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions with $f(x) = f(1 - x)$ for all real x.

      - This set $\boxed{\text{is a subspace}}$ because it contains the zero vector and is closed under addition and scalar multiplication.

   (e) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with integer entries.

      - This set $\boxed{\text{is not a subspace}}$ because it is not closed under scalar multiplication (specifically, by non-integer scalars).

   (f) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with nonnegative real entries.

      - This set $\boxed{\text{is not a subspace}}$ because it is not closed under scalar multiplication (specifically, by negative scalars).

   (g) $V = P_3(\mathbb{C})$, $S =$ the polynomials in V with $p(i) = 0$.

      - This set $\boxed{\text{is a subspace}}$ because it contains the zero polynomial and is closed under addition and scalar multiplication.

   (h) $V = M_{2\times 2}(\mathbb{Q})$, $S =$ the matrices in V of determinant zero.

      - This set $\boxed{\text{is not a subspace}}$ because it is not closed under addition. For example, S contains $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ but not their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (It does satisfy the other two properties, however.)

   (i) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions that are zero at every rational number.

      - This set $\boxed{\text{is a subspace}}$ because it contains the zero function and is closed under addition and scalar multiplication. (Note that V contains lots of functions, such as the function that is 1 at $x = \sqrt{2}$ and 0 everywhere else.)

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3. For each set of vectors in each vector space, determine (i) if they span V and (ii) if they are linearly independent:

   (a) $\langle 1, 2 \rangle$, $\langle 3, 2 \rangle$, $\langle 1, 1 \rangle$ in $\mathbb{R}^2$.

      - Some quick calculations will show that these vectors $\boxed{\text{do span}}$, but are $\boxed{\text{not linearly independent}}$.

   (b) $\langle 1, 2, 4 \rangle$, $\langle 3, 2, 1 \rangle$, $\langle 1, 1, 1 \rangle$ in $\mathbb{R}^3$.

      - Since we have 3 vectors in $\mathbb{R}^3$ we can use the determinant shortcut: the matrix whose columns are the three given vectors has nonzero determinant, so the three vectors $\boxed{\text{do span}}$ and $\boxed{\text{are linearly independent}}$.

(c) $1+x$, $x+x^2$ in $P_2(\mathbb{C})$.

- These 2 polynomials $\boxed{\text{do not span}}$ because $P_2(\mathbb{C})$ has dimension 3. They $\boxed{\text{are linearly independent}}$, however, because neither is a scalar multiple of the other.

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_5)$. [Note $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$; the entries of the matrices are considered modulo 5.]

- These 2 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They $\boxed{\text{are linearly independent}}$, however, because neither is a scalar multiple of the other.

(e) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$.

- These 3 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They $\boxed{\text{are linearly independent}}$, because $a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ yields $a+c = a+b = b+c = 0$ which has only the solution $a = b = c = 0$.

(f) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_2)$.

- These 3 matrices $\boxed{\text{do not span}}$ because $M_{2\times 2}$ has dimension 4. They are $\boxed{\text{not linearly independent}}$ because their sum is the zero matrix.

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4. Suppose A is an $m \times n$ matrix with entries from the field F.

(a) Show that the set of all vectors $\mathbf{x} \in F^n$ such that $A\mathbf{x}$ equals the zero vector (in F^m) is a subspace of F^n.

- We simply check the subspace criterion:
- For [S1], clearly $A\mathbf{0} = \mathbf{0}$.
- For [S2], if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
- For [S3], if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$ then $A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0}$.

(b) Deduce that the set of solutions to any homogeneous system of linear equations (i.e., in which all of the constants are equal to zero) over F is an F-vector space.

- If we take A to be the coefficient matrix, then the variable vector $\mathbf{x}$ is a simultaneous solution to all of the equations if and only if $A\mathbf{x} = \mathbf{0}$.
- So by part (a), the space of solutions is a subspace of F^m hence is a vector space.

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5. Suppose V is a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

(a) If S is linearly independent, show that T is linearly independent.

- Suppose we had a dependence $a(\mathbf{v}_1) + b(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$.
- Distributing yields $(a+b+c)\mathbf{v}_1 + (b+c)\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$, and so linear independence of S requires $a+b+c = b+c = c = 0$, which has only the solution $a = b = c = 0$. Hence T is linearly independent.

(b) If S spans V, show that T spans V.

- If S spans V then for any $\mathbf{w} \in V$ we can write $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$.
- Then it is not hard to see that $\mathbf{w} = (a-b)\mathbf{v}_1 + (b-c)(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$, so T also spans V.

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6. If V is a vector space and W_1, W_2 are two subspaces of V, their <u>sum</u> is defined to be the set $W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1 \text{ and } \mathbf{w}_2 \in W_2\}$ of all sums of an element of W_1 with an element of W_2.

(a) Prove that $W_1 + W_2$ contains W_1 and W_2, and is a subspace of V.

- For the first part, for any $\mathbf{w}_1$ in W_1 and $\mathbf{w}_2$ in W_2 we can write $\mathbf{w}_1 = \mathbf{w}_1 + \mathbf{0}$ and $\mathbf{w}_2 = \mathbf{0} + \mathbf{w}_2$. Thus so $\mathbf{w}_1$ and $\mathbf{w}_2$ are both in $W_1 + W_2$, and so $W_1 + W_2$ contains W_1 and W_2.
- For the other part, we check the subspace criterion.
- For [S1], $\mathbf{0} = \mathbf{0} + \mathbf{0}$ so $W_1 + W_2$ contains $\mathbf{0}$.
- For [S2], suppose $\mathbf{a}_1 + \mathbf{b}_1$ and $\mathbf{a}_2 + \mathbf{b}_2$ are in $W_1 + W_2$. Then $\mathbf{a}_1 + \mathbf{a}_2$ is in W_1 (by the subspace criterion in W_1) and $\mathbf{b}_1 + \mathbf{b}_2$ is in W_2 (by the subspace criterion in W_2). So since $(\mathbf{a}_1 + \mathbf{b}_1) + (\mathbf{a}_2 + \mathbf{b}_2) = (\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{b}_1 + \mathbf{b}_2)$ we conclude that $(\mathbf{a}_1 + \mathbf{b}_1) + (\mathbf{a}_2 + \mathbf{b}_2)$ is in $W_1 + W_2$.
- For [S3], suppose $\mathbf{a} + \mathbf{b}$ is in $W_1 + W_2$. Then $c\mathbf{a}$ is in W_1 and $c\mathbf{b}$ is in W_2 so $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ is in $W_1 + W_2$.

(b) Prove in fact that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2. [Hint: If W is a subspace of V containing W_1 and W_2, show that W must contain $W_1 + W_2$.]

- Suppose W is a subspace of V containing both W_1 and W_2 and let $\mathbf{a} + \mathbf{b}$ be any vector in $W_1 + W_2$.
- Since $\mathbf{a}$ is in W_1 and $\mathbf{b}$ is in W_2, both $\mathbf{a}$ and $\mathbf{b}$ are in W. So by the subspace criterion in W, $\mathbf{a} + \mathbf{b}$ is in W.
- Since $\mathbf{a} + \mathbf{b}$ was an arbitrary element of $W_1 + W_2$, we conclude that $W_1 + W_2$ is contained in W.
- Therefore, every subspace containing W_1 and W_2 contains $W_1 + W_2$. Since $W_1 + W_2$ is itself a subspace by (a), it is the smallest.

(c) For $V = F[x]$, let W_1 be the subspace of all even polynomials (i.e., polynomials with all terms of even degree) and W_2 be the subspace of all odd polynomials (polynomials with all terms of odd degree). Show that $V = W_1 + W_2$.

- Clearly $W_1 + W_2 \subseteq V$.
- Also, if $p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is an arbitrary element of V, then we can write $p = (a_0 + a_2 x^2 + a_4 x^4 + \cdots) + (a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$.
- Since the first polynomial is in W_1 and the second is in W_2, every element in V is in $W_1 + W_2$, so $V = W_1 + W_2$.

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7. Suppose that $f_0, f_1, \ldots, f_n$ are real-valued functions of x, all of which are n times differentiable. The <u>Wronskian</u> $W(f_0, f_1, \ldots, f_n)$ is defined to be the determinant $W(f_0, f_1, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$. For example, $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$ and $W(x^2, 2x^2) = \begin{vmatrix} x^2 & 2x^2 \\ 2x & 4x \end{vmatrix} = 0$.

(a) Show that if $f_0, f_1, \ldots, f_n$ are linearly dependent, then their Wronskian is zero.

- Suppose that $a_0 f_0 + a_1 f_1 + \cdots + a_n f_n = 0$. Then by taking derivatives we also have $a_0 f_0' + a_1 f_1' + \cdots + a_n f_n' = 0$ and similarly for the higher derivatives.
- This means $\begin{bmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{0}$. But as we have shown, if there is a nonzero vector $\mathbf{v}$ with $M\mathbf{v} = \mathbf{0}$, then $\det(M) = 0$: hence the Wronskian is zero as claimed.

(b) Deduce that if functions $f_0, f_1, \ldots, f_n$ have a nonzero Wronskian, then they are linearly independent. Is the converse true? [Hint: No. Try $f_0 = x^2$ and $f_1 = x\,|x|$.]

- The first part is just the contrapositive of part (a).
- For the second part, observe that for $f_0 = x^2$ and $f_1 = x\,|x|$ we have $f_0' = 2x$ and $f_1' = 2\,|x|$ (this is straightforward to check with a graph or the definition of the derivative). Then $W(f_0, f_1) = \begin{vmatrix} x^2 & x\,|x| \\ 2x & 2\,|x| \end{vmatrix} = x^2 \cdot 2\,|x| - 2x \cdot x\,|x| = 0$.
- However, f_0 and f_1 are linearly independent: if $ax^2 + bx\,|x| = 0$ then setting $x = 1$ yields $a + b = 0$ and setting $x = -1$ yields $a - b = 0$, so that $a = b = 0$.

(c) Show that $\{1, \sin x, \cos x\}$ is a linearly independent set.

- We simply compute the Wronskian: it is $W(1, \sin x, \cos x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -\cos^2 x - \sin^2 x = -1$. Since this is nonzero, by (b) we conclude that the functions are linearly independent.

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8. [Challenge] Let D_n denote the value of the $(n-1) \times (n-1)$ determinant $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$.

Determine whether $\lim_{n\to\infty} \dfrac{D_n}{n!}$ exists.

- Subtract the first row from the other rows, yielding $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{vmatrix}$.
- Now subtract 1/3 of the second row, 1/4 of the third row, 1/5 of the fourth row, ... , $1/n$th of the $(n-1)$st row, from the first row.
- This yields $\begin{vmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{vmatrix}$ where $x = 3 + \dfrac{2}{3} + \dfrac{2}{4} + \dfrac{2}{5} + \cdots + \dfrac{2}{n-1}$.
- Now the matrix is lower triangular so its determinant is simply $x \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot n$.
- Then $\dfrac{D_n}{n!} = \dfrac{x}{2} = \dfrac{3}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \cdots + \dfrac{1}{n-1} = \sum_{k=1}^{n-1} \dfrac{1}{k}$. As $n \to \infty$ this series is the harmonic series, which $\boxed{\text{diverges to } \infty}$.
- Remark: This was problem B5 from the 1992 Putnam.
