E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 2, due Fri Jan 24th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:
   (a) Every vector space contains a zero vector.
   (b) In any vector space, $\alpha \mathbf{v} = \alpha \mathbf{w}$ implies that $\mathbf{v} = \mathbf{w}$.
   (c) In any vector space, $\alpha \mathbf{v} = \beta \mathbf{v}$ implies that $\alpha = \beta$.
   (d) If U is a subspace of V and V is a subspace of W, then U is a subspace of W.
   (e) The empty set is a subspace of any vector space.
   (f) The intersection of two subspaces is always a subspace.
   (g) The union of two subspaces is always a subspace.
   (h) The union of two subspaces is never a subspace.
   (i) The span of the empty set is the empty set.
   (j) The span of the zero vector is the zero subspace.
   (k) If S is any subset of V, then $\text{span}(S)$ is the intersection of all subspaces of V containing S.
   (l) If S is any subset of V, then $\text{span}(S)$ always contains the zero vector.
   (m) Any set containing the zero vector is linearly independent.
   (n) Any subset of a linearly independent set is linearly independent.
   (o) Any subset of a linearly dependent set is linearly dependent.

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2. Determine whether or not each given set S is a subspace of the given vector space V. For each set that is not a subspace, identify at least one part of the subspace criterion that fails.
   (a) $V = \mathbb{R}^4$, $S =$ the vectors $\mathbf{v}$ in $\mathbb{R}^4$ with $\mathbf{v} \cdot \langle 1, 0, 1, 1 \rangle = 2$.
   (b) $V =$ real-valued functions on $[0, 1]$, $S =$ the functions with $f''(x) = f(x)$.
   (c) $V = \mathbb{C}^5$, $S =$ the vectors $\langle a, b, c, d, e \rangle$ with $e = a + b + c$ and $b = c = d$.
   (d) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions with $f(x) = f(1 - x)$ for all real x.
   (e) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with integer entries.
   (f) $V = M_{3\times 3}(\mathbb{R})$, $S =$ the 3×3 matrices with nonnegative real entries.
   (g) $V = P_3(\mathbb{C})$, $S =$ the polynomials in V with $p(i) = 0$.
   (h) $V = M_{2\times 2}(\mathbb{Q})$, $S =$ the matrices in V of determinant zero.
   (i) $V =$ real-valued functions on $\mathbb{R}$, $S =$ the functions that are zero at every rational number.

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3. For each set of vectors in each vector space, determine (i) if they span V and (ii) if they are linearly independent:
   (a) $\langle 1, 2 \rangle$, $\langle 3, 2 \rangle$, $\langle 1, 1 \rangle$ in $\mathbb{R}^2$.
   (b) $\langle 1, 2, 4 \rangle$, $\langle 3, 2, 1 \rangle$, $\langle 1, 1, 1 \rangle$ in $\mathbb{R}^3$.
   (c) $1 + x$, $x + x^2$ in $P_2(\mathbb{C})$.
   (d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_5)$. [Note $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$; the entries are considered modulo 5.]
   (e) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$.
   (f) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{F}_2)$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose A is an $m \times n$ matrix with entries from the field F.

   (a) Show that the set of all vectors $\mathbf{x} \in F^n$ such that $A\mathbf{x}$ equals the zero vector (in F^m) is a subspace of F^n.

   (b) Deduce that the set of solutions to any homogeneous system of linear equations (i.e., in which all of the constants are equal to zero) over F is an F-vector space.

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5. Suppose V is a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$.

   (a) If S is linearly independent, show that T is linearly independent.

   (b) If S spans V, show that T spans V.

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6. If V is a vector space and W_1, W_2 are two subspaces of V, their sum is defined to be the set $W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 : \mathbf{w}_1 \in W_1 \text{ and } \mathbf{w}_2 \in W_2\}$ of all sums of an element of W_1 with an element of W_2.

   (a) Prove that $W_1 + W_2$ contains W_1 and W_2, and is a subspace of V.

   (b) Prove in fact that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2. [Hint: If W is a subspace of V containing W_1 and W_2, show that W must contain $W_1 + W_2$.]

   (c) For $V = F[x]$, let W_1 be the subspace of all even polynomials (i.e., polynomials with all terms of even degree) and W_2 be the subspace of all odd polynomials (polynomials with all terms of odd degree). Show that $V = W_1 + W_2$.

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7. Suppose that $f_0, f_1, \ldots, f_n$ are real-valued functions of x, all of which are n times differentiable. The Wronskian $W(f_0, f_1, \ldots, f_n)$ is defined to be the determinant $W(f_0, f_1, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$. For example, $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$ and $W(x^2, 2x^2) = \begin{vmatrix} x^2 & 2x^2 \\ 2x & 4x \end{vmatrix} = 0$.

   (a) Show that if $f_0, f_1, \ldots, f_n$ are linearly dependent, then their Wronskian is zero.

   (b) Deduce that if functions $f_0, f_1, \ldots, f_n$ have a nonzero Wronskian, then they are linearly independent. Is the converse true? [Hint: No. Try $f_0 = x^2$ and $f_1 = x\,|x|$.]

   (c) Show that $\{1, \sin x, \cos x\}$ is a linearly independent set.

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8. [Challenge] Let D_n denote the value of the $(n-1) \times (n-1)$ determinant $\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}$.
   Determine whether $\lim_{n\to\infty} \dfrac{D_n}{n!}$ exists.
