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1. Identify each of the following statements as true or false:

   (a) If $f : A \to A$ is a one-to-one function, then f must be onto.

   - $\boxed{\text{False}}$. There exist one-to-one functions that are not onto, such as $f(n) = n + 1$ on the positive integers.

   (b) The set of integers $\mathbb{Z}$ is not a field.

   - $\boxed{\text{True}}$. Some elements, like 2 and 5, have no multiplicative inverse in $\mathbb{Z}$.

   (c) Every field has infinitely many elements.

   - $\boxed{\text{False}}$. There exist fields with finitely many elements, like $\mathbb{Z}/p\mathbb{Z}$.

   (d) It is impossible to have $6 = 0$ in a field F.

   - $\boxed{\text{False}}$. There exist fields where $6 = 0$, like the fields $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

   (e) There is a system of linear equations over $\mathbb{R}$ having exactly two different solutions.

   - $\boxed{\text{False}}$. Over $\mathbb{R}$ the only possibilities are that there are no solutions, exactly 1 solution, or infinitely many solutions.

   (f) For any $n \times n$ matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$.

   - $\boxed{\text{False}}$. The correct formula would be $(A + B)^2 = A^2 + AB + BA + B^2$, since matrix multiplication is not commutative.

   (g) For any $n \times n$ matrices A and B, $(BA)^T = B^T A^T$.

   - $\boxed{\text{False}}$. The correct formula would be $(BA)^T = A^T B^T$.

   (h) For any invertible $n \times n$ matrices A and B, $(A + B)^{-1} = A^{-1} + B^{-1}$.

   - $\boxed{\text{False}}$. In fact this formula is almost never correct. An explicit counterexample is $A = B = I_n$: then $(A + B)^{-1} = \frac{1}{2} I_n$ while $A^{-1} + B^{-1} = 2I_n$.

   (i) For any invertible $n \times n$ matrices A and B, $(BA)^{-1} = A^{-1}B^{-1}$.

   - $\boxed{\text{True}}$. The inverse of a product is the product of the inverses in reverse order.

   (j) If A and B are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.

   - $\boxed{\text{True}}$. The determinant is multiplicative so $\det(AB) = \det(A)\det(B) = 6$.

   (k) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(2A) = 3n$.

   - $\boxed{\text{False}}$. Doubling a matrix doubles each row, so if there are n rows, the correct formula would be $\det(2A) = 2^n \cdot 3$.

   (l) For any $n \times n$ matrix A, $\det(A) = -\det(A^T)$.

   - $\boxed{\text{False}}$. The determinant of a transpose equals the determinant of the original matrix, so $\det(A) = \det(A^T)$.

   (m) For any $n \times n$ matrices A and B, $\det(AB) = \det(B)\det(A)$.

   - $\boxed{\text{True}}$. The determinant is multiplicative so $\det(AB) = \det(A)\det(B) = \det(B)\det(A)$.

   (n) If the coefficient matrix of a system of 6 linear equations in 6 unknowns is invertible, then the system has infinitely many solutions.

   - $\boxed{\text{False}}$. If the coefficient matrix is invertible, in fact there is a unique solution: if the system is $A\mathbf{x} = \mathbf{c}$, multiplying on the left by A^{-1} yields the solution $\mathbf{x} = A^{-1}\mathbf{c}$.

   (o) If p and q are polynomials in $F[x]$ of the same degree n, then $p + q$ also has degree n.

- $\boxed{\text{False}}$. The degree could be lower: for example, if $p = x^2 + 2$ and $q = -x^2 + 3x$, then $p + q = 3x + 2$ only has degree 1.

(p) If p and q are polynomials in $F[x]$ of the same degree n, then $p \cdot q$ has degree n^2.

- $\boxed{\text{False}}$. In general, $\deg(pq) = \deg(p) + \deg(q)$, so the degree is $2n$, not n^2.

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2. Find the general solution to each system of linear equations:

(a) $\left\{ \begin{array}{c} -x - 3y + 5z = 9 \\ 3x + 2y + 2z = 0 \\ 2x + 2y + 3z = 4 \end{array} \right\}$.

- By row-reducing, the solution is $(x, y, z) = \boxed{(-2, 1, 2)}$.

(b) $\left\{ \begin{array}{c} x - 2y + 4z = 4 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

- By row-reducing the solution is $(x, y, z) = \boxed{(2 - 4z, -1, z)}$.

(c) $\left\{ \begin{array}{lllllll} a + b + c + d & & = 2 \\ a + b + c & + e & = 3 \\ a + b & + d + e & = 4 \\ a & + c + d + e & = 5 \\ & b + c + d + e & = 6 \end{array} \right\}$.

- By row-reducing, the solution is $\boxed{(a, b, c, d, e) = (-1, 0, 1, 2, 3)}$.

(d) $\left\{ \begin{array}{c} x + 3y + z = -4 \\ -x - 6y + 8z = 10 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

- By row-reducing, there is $\boxed{\text{no solution}}$.

(e) $\left\{ \begin{array}{ccc} a + b + c + d + e & = & 1 \\ a + 2b + 3c + 4d + 5e & = & 6 \end{array} \right\}$.

- By row-reducing, the solution is $(a, b, c, d, e) = \boxed{(-4 + c + 2d + 3e, 5 - 2c - 3d - 4e)}$.

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3. Compute the following things:

(a) If $\mathbf{v} = (3, 0, -4)$ and $\mathbf{w} = (-1, 6, 2)$ in $\mathbb{R}^3$, find $\mathbf{v} + 2\mathbf{w}$, $||\mathbf{v}||$, $||\mathbf{w}||$, $||\mathbf{v} + 2\mathbf{w}||$, and $\mathbf{v} \cdot \mathbf{w}$.

- We have $\mathbf{v} + 2\mathbf{w} = \boxed{(1, 12, 0)}$, $||\mathbf{v}|| = \boxed{5}$, $||\mathbf{w}|| = \boxed{\sqrt{41}}$, $||\mathbf{v} + 2\mathbf{w}|| = \boxed{\sqrt{145}}$, and $\mathbf{v} \cdot \mathbf{w} = \boxed{-11}$.

(b) The sum and product of the polynomials $2x + 3$ and $x^2 - 1$ in $\mathbb{R}[x]$.

- The sum is $x^2 + 2x + 2$ and the product is $2x^3 + 3x^2 - 2x - 3$.

(c) The reduced row-echelon forms of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 2 & 3 \\ 2 & 1 & 0 & -1 & -2 \\ -4 & -2 & 0 & 3 & 0 \end{bmatrix}$.

- Row-reducing yields the RREFs $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(d) The determinants of $\begin{bmatrix} -1 & 5 & 2 \\ 0 & -3 & 7 \\ 2 & 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$.

- The determinants are $\det(A) = \boxed{141}$ and $\det(B) = \boxed{12}$.

(e) The inverses of $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -3 & -2 \\ -3 & 7 & 8 \\ 2 & -6 & -5 \end{bmatrix}$.

- The inverses are $\begin{bmatrix} -3 & -3 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ and $\frac{1}{2}\begin{bmatrix} 13 & -3 & -10 \\ 1 & -1 & -2 \\ 4 & 0 & -2 \end{bmatrix}$ respectively.

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4. Let $\mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$.

(a) Prove that $||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2$.

- Note that

$$\begin{aligned} ||\mathbf{v}+\mathbf{w}||^2 &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = ||\mathbf{v}||^2 + 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2 \\ ||\mathbf{v}-\mathbf{w}||^2 &= (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} = ||\mathbf{v}||^2 - 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2 \end{aligned}$$

and so adding the expressions produces $||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2$.

(b) Deduce that in any parallelogram, the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides. [Hint: Suppose the sides are vectors $\mathbf{v}$ and $\mathbf{w}$.]

- Suppose that two sides are represented by the vectors $\mathbf{v}$ and $\mathbf{w}$ emanating from the same vertex.
- From a diagram we can see that the other two sides are also $\mathbf{v}$ and $\mathbf{w}$, while the diagonals are $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ (with appropriate directions).
- Thus, the sum of the squares of the lengths of the diagonals is $||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2$, while the sum of the squares of the lengths of the four sides is $2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2$. By part (a), these are equal.

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5. Suppose that A and B are $n \times n$ matrices with entries from a field F.

(a) If AB is invertible, show that A and B are invertible.

- Notice that $AB(AB)^{-1} = I_n$, and so $B(AB)^{-1}$ is a right inverse of the matrix A. This means A is invertible.
- Likewise, $(AB)^{-1}AB = I_n$, so $(AB)^{-1}A$ is a left inverse of the matrix B. This means B is invertible.
- Alternatively, since AB is invertible, $\det(AB) = \det(A)\det(B)$ is nonzero. This can only happen when $\det(A)$ and $\det(B)$ are both nonzero, which is to say, when A and B are both invertible.

(b) If A is invertible, show that A^T is invertible and that its inverse is $(A^{-1})^T$.

- Since $\det(A^T) = \det(A)$, if A is invertible then A^T will also be invertible.
- Furthermore, by using the fact that $A^T B^T = (BA)^T$ with $B = A^{-1}$, we see that $A^T(A^{-1})^T = (A^{-1}A)^T = (I_n)^T = I_n$.
- In the same way, $(A^{-1})^T A^T = (AA^{-1})^T = (I_n)^T = I_n$, and so $(A^{-1})^T$ satisfies the inverse matrix property for A^T: this means $(A^T)^{-1} = (A^{-1})^T$.

(c) If $I_n + AB$ is invertible, show that $I_n + BA$ is also invertible. [Hint: Consider $M = I_n - B(I_n + AB)^{-1}A$.]

- Let $M = I_n - B(I_n + AB)^{-1}A$. Then $M(I_n + BA) = (I_n + BA) - B(I_n + AB)^{-1}A(I_n + BA) = (I_n + BA) - B(I_n + AB)^{-1}(A + ABA) = (I_n + BA) - B(I_n + AB)^{-1}(I_n + AB)A = (I_n + BA) - BA = I_n$ and therefore M is the inverse of $I_n + BA$.

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6. Let F be a field of characteristic not 2 (i.e., in which $2 \neq 0$). A square matrix A with entries from F is called symmetric if $A = A^T$ and skew-symmetric if $A = -A^T$.

   (a) For any $n \times n$ matrix B, show that $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

   - Observe that $(B + B^T)^T = B^T + (B^T)^T = B^T + B$ so this matrix equals its transpose hence is symmetric.
   - Similarly, $(B - B^T)^T = B^T - B$, so this matrix is -1 times its transpose hence is skew-symmetric.

   (b) Show that any square matrix M can be written *uniquely* in the form $M = S + T$ where S is symmetric and T is skew-symmetric. [Make sure to prove that there is *only* one such decomposition!]

   - If $M = S + T$ then $M^T = S^T + T^T = S - T$. Solving for S, T produces $S = \frac{1}{2}(M + M^T)$ and $T = \frac{1}{2}(M - M^T)$, so this is the only possible solution. (Here we are using the fact that $2 \neq 0$, so we can divide by 2.)
   - By part (a), we see $S = \frac{1}{2}(M + M^T)$ is symmetric and $T = \frac{1}{2}(M - M^T)$ is skew-symmetric, so these choices do work. Hence there is a unique decomposition as claimed.

   (c) If A is a skew-symmetric $n \times n$ real matrix and n is odd, show that $\det(A) = 0$.

   - Taking the determinant of both sides of $\det(A) = \det(-A^T)$ yields $\det(A) = (-1)^n \det(A^T) = (-1)^n \det(A)$.
   - Since n is odd, this gives $\det(A) = -\det(A)$, meaning $\det(A) = 0$ since $2 \neq 0$ (and thus $1 \neq -1$).

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7. Prove the following things via induction (or otherwise):

   (a) The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. (Thus $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so forth.) Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for every positive integer n.

   - We use induction on n. For the base case $n = 1$, we have $F_1 = 1 = F_2$ which is true.
   - For the inductive step suppose that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$. Then $F_1 + F_3 + F_5 + \cdots + F_{2n-1} + F_{2n+1} = [F_1 + F_3 + F_5 + \cdots + F_{2n-1}] + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2}$ as required.

   (b) Prove that the nth power of the matrix $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix}$ for each positive integer n.

   - Induction on n. The base case $n = 1$ follows as $\begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ for $n = 1$.
   - For the inductive step, suppose $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^n = \begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix}$. Then $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^{n+1}$ $= \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}^n = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1-2n & 4n \\ -n & 1+2n \end{bmatrix} = \begin{bmatrix} 1-2(n+1) & 4(n+1) \\ -(n+1) & 1+2(n+1) \end{bmatrix}$.

   (c) Let M_n be the $n \times n$ matrix with 1s on the diagonal and directly below the diagonal, -1s directly above the diagonal, and 0s elsewhere. Prove that $\det(M_n)$ is the $(n+1)$st Fibonacci number F_{n+1}.

   - We use strong induction on n. The base cases $n = 1$ and $n = 2$ follow by observing that $M_1 = [1]$ so $\det(M_1) = 1 = F_2$ and that $M_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ so $\det(M_2) = 2 = F_3$, as required.
   - For the inductive step, suppose $\det(M_{n-1}) = F_n$ and $\det(M_{n-2}) = F_{n-1}$ and consider expanding $\det(M_n)$ along the first row. Only the terms from the first and second entries contribute, since all other entries in the first row are zero. Deleting the first row and column of M_n yields M_{n-1}, while deleting the first row and second column of M_n yields a matrix whose first column has a 1 and then all zeroes, so its determinant is the same as the determinant obtained by deleting its first row and column, which results in the matrix M_{n-2}.
   - Thus via expansion by minors we see $\det(M_n) = 1 \cdot \det(M_{n-1}) - (-1) \cdot \det(M_{n-2}) = F_n + F_{n-1} = F_{n+1}$.

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8. [Challenge] Let F be a field and suppose $x_1, \dots, x_n$ are elements of F. The goal of this problem is to evaluate the famous <u>Vandermonde determinant</u> $V(x_1, \dots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$.

(a) Show that if any of the x_i are equal to one another, then $V(x_1, \dots, x_n) = 0$.

- If $x_i = x_j$ then the ith and jth rows of the matrix are equal, so its determinant is zero.

(b) Show that as a polynomial in the variables $x_1, \dots, x_n$, $V(x_1, \dots, x_n)$ has degree $\frac{n(n-1)}{2}$ and is divisible by $x_j - x_i$ for any $i \neq j$. [Hint: Use (a) and the remainder theorem.]

- For the degree, observe that when we expand the determinant by minors, each term will be a product of one term from the first column, one from the second column, ... , and one from the last column, so the resulting product will have degree $0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$.
- Now dividing $V(x_1, \dots, x_n)$ by $x_i - x_j$ (where we think of x_i as the variable) leaves some remainder term. When we set $x_i = x_j$ then the remainder term vanishes by part (a), so the remainder must be the zero polynomial.

(c) Deduce that $V(x_1, \dots, x_n)$ is divisible by the product $\prod_{1 \le i < j \le n}(x_j - x_i)$ and that this product is a polynomial of degree $\frac{n(n-1)}{2}$.

- By (b) applied to all possible pairs (i, j) with $1 \le i < j \le n$ we see that $x_j - x_i$ divides $V(x_1, \dots, x_n)$. Since these terms are all relatively prime, their product must divide $V(x_1, \dots, x_n)$.
- Furthermore, the number of possible pairs (i, j) is simply $\binom{n}{2} = \frac{n(n-1)}{2}$ since we may pick any unordered pair of values $\{x_i, x_j\}$, so the product of these terms has degree $\frac{n(n-1)}{2}$.

(d) Show in fact that $V(x_1, \dots, x_n) = \prod_{1 \le i < j \le n}(x_j - x_i)$. [Hint: Compare degrees and coefficients of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.]

- By (b) and (c) we see that dividing V by the product yields a polynomial of degree 0 (in other words, a constant). But since the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in V is equal to 1 (it comes from the product of terms on the diagonal of the matrix) and the coefficient in the product is also equal to 1 (it comes from the product of the all the first terms x_j with $j > i$ in each pair), the constant must equal 1.
- Thus, $V(x_1, \dots, x_n) = \prod_{1 \le i < j \le n}(x_j - x_i)$ as claimed.

(e) Suppose that $x_1, \dots, x_n \in F$ are distinct and $y_1, \dots, y_n \in F$ are arbitrary. Prove that there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ in $F[x]$ of degree at most $n-1$ such that $p(x_i) = y_i$ for each $1 \le i \le n$. [Hint: Write down the corresponding system of linear equations.]

- We have the equations $a_0 + a_1 x_1 + \cdots + a_{n-1}x_1^{n-1} = y_1$, ... , $a_0 + a_1 x_n + \cdots + a_{n-1}x_n^{n-1} = y_n$, which in matrix form is $\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$.
- The coefficient matrix is precisely the Vandermonde matrix we have been analyzing. By the formula in part (d), its determinant is nonzero (as all of the x_i are distinct) and therefore it is invertible, so the system has a unique solution.
- This means there is a unique solution to the system, which is to say, there is a unique polynomial $p(x)$ with the desired properties.
