E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 12, due Fri Apr 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V and W be finite-dimensional real or complex vector spaces and T be linear.
   (a) The singular values of $T : V \to V$ are the absolute values of the eigenvalues of T.
   (b) If T is Hermitian, the singular values of $T : V \to V$ are absolute values of the eigenvalues of T.
   (c) The singular value decomposition of a matrix is unique.
   (d) The pseudoinverse of a matrix is unique.
   (e) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $TT^\dagger(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)$.
   (f) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $TT^\dagger(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)^\perp$.
   (g) If $T : V \to V$ is an isomorphism, then $T^\dagger = T^{-1}$.
   (h) For any inconsistent system $A\mathbf{x} = \mathbf{c}$, the vector $\hat{\mathbf{x}} = A^\dagger \mathbf{c}$ is the system's unique least-squares solution.
   (i) If $T : V \to W$ is linear and T^*T is invertible, then $T^\dagger = (T^*T)^{-1}T^*$.

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2. For each matrix M, find (i) the singular values of M, (ii) a singular value decomposition $M = U\Sigma V^*$ where U and V are unitary and Σ is a rectangular diagonal matrix, and (iii) the pseudoinverse $M^\dagger$ of M:

   (a) $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$. (c) $\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix}$. (d) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$. (e) $\begin{bmatrix} 1 & i & -1 & -i \\ 2 & 2 & 2 & 2 \end{bmatrix}$.

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3. Let $A = \begin{bmatrix} 2 & -8 & 2 \\ 6 & 6 & -9 \end{bmatrix}$.
   (a) Find singular value decompositions for A and for A^T.
   (b) Find the pseudoinverses $A^\dagger$ and $(A^T)^\dagger$.
   (c) Find the solution $\mathbf{x}$ to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ of minimal norm.
   (d) Find the least-squares solution to the inconsistent system $A^T\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. A Hermitian matrix A is said to be positive-definite if $\mathbf{v}^*A\mathbf{v} > 0$ for every $\mathbf{v} \neq \mathbf{0}$. The goal of this problem is to prove Sylvester's criterion for positive-definiteness: if A is an $n \times n$ Hermitian matrix, then A is positive definite if and only if $\det A^{(k)} > 0$ for all $1 \le k \le n$, where $A^{(k)}$ is the upper $k \times k$ submatrix of A. So suppose $A \in M_{n\times n}(\mathbb{C})$ is Hermitian.
   (a) If A is positive definite, show that $A^{(k)}$ is positive definite for each $1 \le k \le n$ and deduce that $\det A^{(k)} > 0$ for all $1 \le k \le n$.
   (b) Suppose that A has two orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ whose eigenvalues λ_1, λ_2 are negative. Show that $A^{(k-1)}$ is not positive definite. [Hint: Show that there exists a linear combination $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ whose last coordinate is zero, and then that $\mathbf{w}^*A\mathbf{w} < 0$.]
   (c) Deduce that if $A^{(k-1)}$ is positive definite and $\det(A) > 0$, then all eigenvalues of A must be positive and hence A is positive definite.
   (d) Suppose that $\det A^{(k)} > 0$ for all $1 \le k \le n$. Show that A is positive definite.

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5. Let S be an $n \times n$ real symmetric matrix.

   (a) Show that S is congruent to a matrix whose diagonal entries are all in the set $\{-1, 0, 1\}$.

   (b) Prove that, up to congruence, there are exactly $\frac{1}{2}(n+1)(n+2)$ different real $n \times n$ symmetric matrices.

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6. By the singular value decomposition theorem, if $T : V \to W$ is a linear transformation of rank r, then there exist orthonormal bases $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of W along with scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ with $T(\mathbf{v}_i) = \sigma_i \mathbf{w}_i$ for $1 \leq i \leq r$ and $T(\mathbf{v}_i) = \mathbf{0}$ for $i > r$.

   (a) Show that $T^*(\mathbf{w}_i) = \sigma_i \mathbf{v}_i$ for $1 \leq i \leq r$ and $T(\mathbf{w}_i) = \mathbf{0}$ for $i > r$. [Hint: Consider $[T]_\beta^\gamma$ and $[T^*]_\gamma^\beta$.]

   (b) Show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of eigenvectors for T^*T with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$, and that $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is a set of eigenvectors for TT^* with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$.

   (c) Deduce that the nonzero eigenvalues of T^*T and TT^* are the same, and hence that the nonzero singular values of T and T^* are the same.

   (d) Show that if $A \in M_{m \times n}(\mathbb{C})$, then the singular values of A and A^* are the same, and that if A^* has a singular value decomposition $A = U\Sigma V^*$ then A^* has a singular value decomposition $A^* = V\Sigma^T U^*$.

   Remark: The results of this problem are useful in computing the SVD of a non-square matrix, since one may just find the nonzero eigenvalues and eigenvectors of the smaller of A^*A and $AA^*$to construct $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$, and then compute $\ker(A)$ to get $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ and $\ker(A^*)$ to get $\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_m\}$.

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7. Suppose $A \in M_{m \times n}(F)$ where $F = \mathbb{R}$ or $\mathbb{C}$.

   (a) Show that $(A^\dagger)^* = (A^*)^\dagger$.

   (b) Show that $AA^\dagger$ and $A^\dagger A$ are positive-semidefinite and Hermitian.

   (c) For $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, show that $(BC)^\dagger \neq C^\dagger B^\dagger$.

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8. [Challenge] The goal of this problem is to discuss matrix square roots and the matrix analogue of the polar form $z = e^{i\theta} r$ of a complex number. Let $A \in M_{n \times n}(\mathbb{C})$.

   (a) Show that there exists a unitary matrix W and a positive semidefinite Hermitian matrix P such that $A = WP$; this is called a (right) polar decomposition of A with W being the analogue of $e^{i\theta}$ and P being the analogue of r. [Hint: Take $W = UV^*$ and $P = V\Sigma V^*$.]

   (b) Show that if B is a positive-semidefinite Hermitian matrix such that $B^2 = \mu I_n$ for some nonnegative scalar μ, then $B = \sqrt{\mu} I_n$.

   (c) Show that if A is a positive-semidefinite Hermitian matrix, then there exists a unique positive-semidefinite Hermitian matrix B satisfying $B^2 = A$ (i.e., a "square root" of A). [Hint: Reduce to the case where A is diagonal, and then use part (b) along with 6(a) from homework 8 on each eigenspace of A.]

   (d) Suppose P and Q are positive-semidefinite Hermitian matrices and $P^2 = Q^2$. Show that $P = Q$.

   (e) Show that the polar decomposition of an invertible matrix A is unique. [Hint: Show first that P is invertible and then that $WP = ZQ$ implies $P^2 = Q^2$.]

   Remark: The usual procedure for finding the polar form of a complex number $z = e^{i\theta} r$ is to note that $r = \sqrt{|z|^2} = \sqrt{\bar{z}z}$ and then $e^{i\theta} = z/r$. For the polar decomposition $A = WP$ we have an analogous formula: $P = \sqrt{A^*A}$, where the square root here denotes the positive-semidefinite matrix square root of (c), and when P is positive-definite we obtain the unitary part W via $W = AP^{-1}$.
