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1. Identify each of the following statements as true or false:
   (a) Every real Hermitian matrix is diagonalizable.
      - True: by the spectral theorem, Hermitian matrices are diagonalizable.

   (b) Every real symmetric matrix is diagonalizable.
      - True: real symmetric matrices are Hermitian, so they are diagonalizable.

   (c) Every complex Hermitian matrix is diagonalizable.
      - True: again by the spectral theorem, Hermitian matrices are diagonalizable.

   (d) Every complex symmetric matrix is diagonalizable.
      - False: $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ is not diagonalizable: its Jordan form has a 2×2 block with eigenvalue 0.

   (e) If the sum of the entries in all columns of the $n \times n$ matrix A equals 1, then 1 is an eigenvalue of A.
      - True: if $\mathbf{v}$ is the vector of all 1s, then $A^T\mathbf{v} = \mathbf{v}$, so 1 is an eigenvalue of A^T and hence of A.

   (f) If the sum of the entries in all columns of a square matrix A with nonnegative real entries equals 1, then $\lim_{n\to\infty} A^n$ exists.
      - False: for example, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then the powers of A alternate between A and I_2.

   (g) If the sum of the entries in all columns of a square matrix A with positive real entries equals 1, then $\lim_{n\to\infty} A^n$ exists.
      - True: this is a theorem mentioned in class about the convergence of powers of stochastic matrices.

   (h) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.
      - True: it is linear in both components, so it is a bilinear form.

   (i) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.
      - False: it is not linear in the second component, so it is not a bilinear form.

   (j) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.
      - False: although this is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$, it is not a bilinear form because it doesn't respect addition or scalar multiplication in the individual components. For example, $\Phi(1, 1) + \Phi(2, 1) \neq \Phi(3, 1)$.

   (k) If $V = F^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \det(\mathbf{v}, \mathbf{w})$, the determinant of the matrix whose columns are $\mathbf{v}$ and $\mathbf{w}$, then Φ is a bilinear form on V.
      - True: by properties of determinants (or simply by writing it out: $\Phi((a, b), (c, d)) = ad - bc$), we can see it is linear in both functions, so it is a bilinear form.

   (l) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \text{tr}(AB)$, then Φ is a bilinear form on V.
      - True: Φ is linear in both A and B, so it is a bilinear form on V. (Indeed, when $F = \mathbb{R}$, this is the Frobenius inner product.)

   (m) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \det(AB)$, then Φ is a bilinear form on V.
      - False: note that $\Phi(A, B) = \det(A)\det(B)$, but the determinant is not a linear function on matrices, so Φ is not linear in either component.

   (n) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 x f(x)g(x)\, dx$, then Φ is a bilinear form on V.
      - True: it is linear in both functions, so it is a bilinear form.

   (o) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 f'(x)g'(x)\, dx$, then Φ is a bilinear form on V.
      - True: derivatives are linear, so it is still linear in both functions and thus a bilinear form.

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2. Solve the following problems:

(a) Find a formula for the nth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.

- We diagonalize this matrix. The characteristic polynomial is $p(t) = t^2 - 5t - 6 = (t-6)(t+1)$ so the eigenvalues are $\lambda = 6, -1$.
- We can compute that $(1,1)$ is a basis for the 6-eigenspace and $(5,-2)$ is a basis for the -1-eigenspace, so if we take $Q = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}$, $Q^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$, then $Q^{-1}AQ = D$.
- We then have $A^n = QD^nQ^{-1} = \boxed{\frac{1}{7}\begin{bmatrix} 2\cdot 6^n + 5(-1)^n & 5\cdot 6^n - 5(-1)^n \\ 2\cdot 6^n - 2(-1)^n & 5\cdot 6^n + 2\cdot(-1)^n \end{bmatrix}}$.

(b) In Diagonalizistan there are two cities: City A and City B. Each year, 2/5 of the residents of City A move to City B, and 2/3 of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n and determine what happens as $n \to \infty$.

- Let a_n be the population in city A in year n and b_n be the population in city B in year n.
- Then the given information implies that $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 3/5 & 2/3 \\ 2/5 & 1/3 \end{bmatrix}^n \begin{bmatrix} 2000 \\ 6000 \end{bmatrix}$.
- To compute the matrix power, we diagonalize $A = \begin{bmatrix} 3/5 & 2/3 \\ 2/5 & 1/3 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = t^2 - \frac{14}{15}t - \frac{1}{15} = (t-1)(t+\frac{1}{15})$, so the eigenvalues are 1 and $-\frac{1}{15}$, with respective eigenspaces spanned by $\begin{bmatrix} 5/3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- So with $Q = \begin{bmatrix} 5/3 & -1 \\ 1 & 1 \end{bmatrix}$ we get $A^n = Q\begin{bmatrix} 1 & 0 \\ 0 & (-1/15)^n \end{bmatrix}Q^{-1} = \frac{1}{8}\begin{bmatrix} 5+3(-1/15)^n & 5-5(-1/15)^n \\ 3-3(-1/15)^n & 3+5(-1/15)^n \end{bmatrix}$.
- Then $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A^n \begin{bmatrix} 2000 \\ 6000 \end{bmatrix} = \begin{bmatrix} 5000 - 3000(-1/15)^n \\ 3000 + 3000(-1/15)^n \end{bmatrix}$. Thus as $n \to \infty$, the populations approach 5000 in city A and 3000 in city B.

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3. Solve each system of differential equations:

(a) Find the general solution to $y_1' = 7y_1 + y_2$ and $y_2' = 9y_1 - y_2$.

- The coefficient matrix is $A = \begin{bmatrix} 7 & 1 \\ 9 & -1 \end{bmatrix}$ with characteristic polynomial is $p(t) = \det(tI - A) = (t-8)(t+2)$, so the eigenvalues are $\lambda = -2, 8$.
- For $\lambda = 8$, the eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- For $\lambda = -2$ the eigenspace is also 1-dimensional and spanned by $\begin{bmatrix} -1 \\ 9 \end{bmatrix}$.
- By the eigenvalue method, the general solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8x} + C_2 \begin{bmatrix} -1 \\ 9 \end{bmatrix} e^{-2x}}$.

(b) Find the general solution to $y_1' = 3y_1 - 2y_2$ and $y_2' = y_1 + y_2$.

- The coefficient matrix is $A = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ with characteristic polynomial is $p(t) = \det(tI - A) = t^2 - 4t + 5$. By the quadratic formula, the eigenvalues are $\lambda = 2 \pm i$.
- For $\lambda = 2 + i$, the eigenspace is 1-dimensional and spanned by $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.
- For $\lambda = 2 - i$ we can take the complex conjugate to get the eigenvector $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

- The general solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+i)x} + C_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-i)x}}$.
- With real functions we get $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{C_1 e^{2x} \begin{bmatrix} \cos(x) - \sin(x) \\ \cos(x) \end{bmatrix} + C_2 e^{2x} \begin{bmatrix} \sin(x) + \cos(x) \\ \sin(x) \end{bmatrix}}$.

(c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]

- Following the hint, if we let $z = y'$ then $z' = y'' = 4y$, so we obtain the system $\left\{ \begin{array}{rcl} y' & = & z \\ z' & = & 4y \end{array} \right\}$ with associated matrix $A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$.
- The characteristic polynomial is $p(t) = t^2 - 4$ with roots $\lambda = -2, 2$.
- The -2 and 2-eigenspaces are spanned by $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively.
- Thus by the eigenvalue method, the general solution is $\begin{bmatrix} y \\ z \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2x} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2x}$.
- In particular, we see $y = \boxed{C_1 e^{-2x} + C_2 e^{2x}}$.

(d) Find the general solution to $y_1' = 2y_2 + \sec(2x)$ and $y_2' = -2y_1$.

- The coefficient matrix for the homogeneous system is $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \pm 2i$. Row-reducing to find eigenvectors yields the complex-valued solution basis $\begin{bmatrix} -i \\ 1 \end{bmatrix} e^{2ix}$, $\begin{bmatrix} i \\ 1 \end{bmatrix} e^{-2ix}$ with equivalent real-valued solution basis $\begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix}, \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}$.
- We want $\tilde{\mathbf{y}} = c_1(x) \begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix} + c_2(x) \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}$ where $\begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(2x) \end{bmatrix}$.
- Left-multiplying by $\begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix}$ yields $\begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} \sin(2x) & -\cos(2x) \\ \cos(2x) & \sin(2x) \end{bmatrix} \begin{bmatrix} 0 \\ \sec(2x) \end{bmatrix} = \begin{bmatrix} -1 \\ \tan(2x) \end{bmatrix}$ and now taking antiderivatives yields $c_1(x) = C_1 - x$ and $c_2(x) = C_2 + \frac{1}{2} \ln(\sec(2x))$.
- The solution is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{(C_2 + \frac{1}{2} \ln(\sec 2x)) \begin{bmatrix} \sin(2x) \\ \cos(2x) \end{bmatrix} + (C_1 - x) \begin{bmatrix} -\cos(2x) \\ \sin(2x) \end{bmatrix}}$.

(e) Solve the system $\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$, where $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

- The coefficient matrix is $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, which is already in Jordan canonical form. Using the matrix exponential formula, we compute $e^{Ax} = \begin{bmatrix} e^{2x} & xe^{2x} & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix}$.
- Then desired solution is $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = e^{Ax}\mathbf{y}(0) = \begin{bmatrix} e^{2x} & xe^{2x} & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2e^{2x} + 3xe^{2x} \\ 3e^{2x} \\ -e^{3x} \end{bmatrix}}$.

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4. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:

(a) The pairing $\Phi((a, b, c), (d, e, f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with β the standard basis.

- We evaluate $\Phi(\beta_i, \beta_j)$ to compute the (i,j)-entry, where $\beta_1 = (1,0,0)$, $\beta_2 = (0,1,0)$, $\beta_3 = (0,0,1)$.
- For example, $\Phi(\beta_1, \beta_2) = \Phi((1,0,0),(0,1,0)) = 1$, while $\Phi(\beta_2, \beta_2) = \Phi((0,1,0),(0,1,0)) = -2$.
- The end result is $\boxed{[\Phi]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}}$

(b) The pairing $\Phi(p,q) = p(-1)q(2)$ on $V = P_3(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.

- We simply evaluate $\Phi(\beta_i, \beta_j)$ to compute the (i,j)-entry, where $\beta_1 = 1$, $\beta_2 = x$, $\beta_3 = x^2$, $\beta_4 = x^3$.
- For example, $\Phi(\beta_1, \beta_2) = 1 \cdot 2 = 2$, while $\Phi(\beta_2, \beta_2) = (-1) \cdot 2 = -2$ and $\Phi(\beta_4, \beta_3) = (-1)^3 2^2 = -4$.
- The end result is $\boxed{[\Phi]_\beta = \begin{bmatrix} 1 & 2 & 4 & 8 \\ -1 & -2 & -4 & -8 \\ 1 & 2 & 4 & 8 \\ -1 & -2 & -4 & -8 \end{bmatrix}}$.

(c) The pairing $\Phi(A,B) = \mathrm{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

- As above we just compute the 16 possible values $\Phi(\beta_i, \beta_j)$ for basis elements β_i, β_i.
- For example, $\Phi(\beta_1, \beta_2) = \mathrm{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = 0$, $\Phi(\beta_2, \beta_3) = \mathrm{tr}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = 1$.
- The end result is $\boxed{[\Phi]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$.

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5. The goal of this problem is to give two proofs of Binet's formula for the Fibonacci numbers defined by the recurrence $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$; the next few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Explicitly, for $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$, Binet's formula says that $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

(a) Show that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and deduce that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- Note that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ by the recurrence.
- Thus, by a trivial induction, we see that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

(b) Find a formula for the nth power of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and use the result to deduce Binet's formula.

- We diagonalize this matrix. The characteristic polynomial is $p(t) = t^2 - t - 1$ so the eigenvalues are $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$.
- Letting $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$, we can compute that $(\varphi, 1)$ is a basis for the φ-eigenspace and so $(\overline{\varphi}, 1)$ is a basis for the $\overline{\varphi}$-eigenspace, so if we take $Q = \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix}$, $Q^{-1} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{bmatrix}$, $D = \begin{bmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{bmatrix}$, then $A = QDQ^{-1}$.
- We then have $A^n = QD^nQ^{-1} = \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \overline{\varphi}^n \end{bmatrix} \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\overline{\varphi} \\ -1 & \varphi \end{bmatrix} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & \varphi^{n-1} \\ -\overline{\varphi}^n & -\overline{\varphi}^{n-1} \end{bmatrix} =$ $\boxed{\dfrac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - \overline{\varphi}^{n+1} & \varphi^n - \overline{\varphi}^n \\ \varphi^n - \overline{\varphi}^n & \varphi^{n-1} - \overline{\varphi}^{n-1} \end{bmatrix}}$.

- Then by (a), we see $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1}+\overline{\varphi}^{n+1} & -\varphi^{n-1}-\overline{\varphi}^{n+1} \\ -\varphi^{n+1}-\overline{\varphi}^{n-1} & \varphi^{n-1}-\overline{\varphi}^{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1}-\overline{\varphi}^{n+1} \\ \varphi^{n}-\overline{\varphi}^{n} \end{bmatrix}$, yielding $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$ as claimed.

(c) Let W be the space of all real sequences $\{a_n\}_{n\geq 0}$ such that $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that W is a 2-dimensional vector space over $\mathbb{R}$.

- To show W is a vector space, simply verify the subspace criterion:
- [S1] W contains the zero sequence.
- [S2] If $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are in W, then for $c_n = a_n + b_n$ we have $c_{n+1} = a_{n+1} + b_{n+1} = (a_n + a_{n-1}) + (b_n + b_{n-1}) = c_n + c_{n-1}$ so $\{c_n\}_{n\geq 0}$ is in W.
- [S3] If $\{a_n\}_{n\geq 0}$ is in W, then for $d_n = ra_n$ we have $d_{n+1} = r(a_n + a_{n-1}) = d_n + d_{n-1}$ so $\{d_n\}_{n\geq 0}$ is in W.
- Furthermore, any such sequence is completely characterized by its 0th and 1st terms by the recurrence, and these values can be chosen freely. Thus, the map $T : W \to \mathbb{R}^2$ with $T(\{a_n\}_{n\geq 0}) = (a_0, a_1)$ is an isomorphism, and so W is 2-dimensional.

(d) With notation as in (c), show that the sequences $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are a basis for W. Deduce that there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$ and then deduce Binet's formula.

- Note that φ and $\overline{\varphi}$ are the two roots of the quadratic $x^2 - x - 1 = 0$ as calculated in (a): thus, $\varphi^2 = \varphi + 1$ and $\overline{\varphi}^2 = \overline{\varphi} + 1$, so multiplying by φ^{n-1} and $\overline{\varphi}^{n-1}$ respectively yields $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$ and $\overline{\varphi}^{n+1} = \overline{\varphi}^n + \overline{\varphi}^{n-1}$.
- Thus, $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are both elements of W. Since they are linearly independent (since both sequences start with 1 at $n = 0$ but are different for $n = 1$) and W is 2-dimensional, they are a basis.
- Thus, there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$.
- We can compute them by setting $n = 0, 1$ to see $0 = C + D$ and $1 = C\varphi + D\overline{\varphi}$, which yields $C = \dfrac{1}{\varphi - \overline{\varphi}} = \dfrac{1}{\sqrt{5}}$ and $D = -C = -\dfrac{1}{\sqrt{5}}$, again giving Binet's formula $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

Remark: Both of these methods extend generally to solve general linear recurrences of the form $a_{n+1} = C_1 a_n + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ for constants $C_1, \ldots, C_k$. Additionally, the matrix formula in (a) is a good source of other Fibonacci identities.

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6. Suppose V is finite-dimensional with scalar field F and $T : V \to V$ is linear. We say the polynomial $q(x) \in F[x]$ annihilates T if $q(T) = 0$.

(a) Show that the set of polynomials in $F[x]$ annihilating T is a vector space.

- Simply check the subspace criterion:
- [S1] Clearly the zero polynomial annihilates T for any T.
- [S2] If $p(x)$ and $q(x)$ annihilate T, then $(p+q)(T) = p(T) + q(T) = 0 + 0 = 0$ so $p + q$ also annihilates T.
- [S3] If $p(x)$ annihilates T, then $(\alpha p)(T) = \alpha p(T) = \alpha 0 = 0$ so αp also annihilates T.

We define the minimal polynomial of T to be the monic polynomial $m(t) \in F[t]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(t) = t - 1$.

(b) Show that every polynomial that annihilates T is divisible by the minimal polynomial. [Hint: Use polynomial division.]

- Suppose $a(t)$ annihilates T and write $a(t) = q(t)m(t) + r(t)$ with $\deg r < \deg m$.
- Then $r(T) = a(T) - q(T)m(T) = 0$ since $a(T) = m(T) = 0$.
- Since $\deg r < \deg m$ and m has minimal positive degree, we must have $r = 0$.

(c) Conclude that the minimal polynomial divides the characteristic polynomial.

- The Cayley-Hamilton theorem says that the characteristic polynomial annihilates T, so by (b), it is divisible by the minimal polynomial.

(d) Suppose λ is an eigenvalue of T. Prove that λ is a root of the minimal polynomial of T, and deduce that the minimal polynomial and the characteristic polynomial have the same set of roots. [Hint: Consider the Jordan form of an associated matrix A.]

- Consider the Jordan canonical form of any associated matrix A to T. Since λ is an eigenvalue of A, it appears on the diagonal of the Jordan form. Then the corresponding diagonal entry of $m(J)$ will be $m(\lambda)$.
- But since $m(J) = m(PAP^{-1}) = p \cdot m(A) \cdot P^{-1} = 0$, this means $m(\lambda) = 0$ so λ is a root of m.
- Since the roots of the characteristic polynomial are the eigenvalues, all its roots are roots of m, and since m divides p, all its roots are roots of p. Thus m and p have the same roots.

(e) Parts (c) and (d) gives a moderately effective way to find the minimal polynomial, namely, test divisors of the characteristic polynomial that have all of the same roots (though not necessarily the same multiplicities). Using this method or otherwise, find the minimal polynomials of the matrices $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.

- $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$: The characteristic polynomial is $(x-1)^2$ and $x-1$ does not annihilate this matrix, so the minimal polynomial must be $(x-1)^2$.
- $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$: The characteristic polynomial is $(x-1)(x-2)^2$ and $(x-1)(x-2)$ does not annihilate this matrix, so the minimal polynomial must be $(x-1)(x-2)^2$.
- $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$: The characteristic polynomial is $(x-1)(x-2)^2$ and $(x-1)(x-2)$ annihilates this matrix, so the minimal polynomial must be $(x-1)(x-2)$.

(f) Show that similar matrices have the same minimal polynomial.

- If $p(A) = 0$ then conjugating yields $p(PAP^{-1}) = 0$, and conversely if $p(PAP^{-1}) = 0$ then conjugating by P^{-1} yields $p(A) = 0$.
- Thus the polynomials annihilating A and PAP^{-1} are the same, and so the minimal polynomials are also the same.

(g) Show that the minimal polynomial of the $k \times k$ Jordan block with eigenvalue λ is $m(t) = (t-\lambda)^k$.

- Note that $J - \lambda I_k$ is the matrix N with 1s directly above the diagonal.
- It follows by a simple induction that N^{k-1} is not the zero matrix (it is the matrix with a single 1 in the upper right corner) but N^k is.
- Thus, $(t-\lambda)^{k-1}$ does not annihilate J, so by (c) and (d) the minimal polynomial must be $(t-\lambda)^k$.

(h) Show that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of A.

- By (f), the minimal polynomial of a $k \times k$ Jordan block is $(t-\lambda)^k$.
- By (g), the minimal polynomial of A is the same as the minimal polynomial of its Jordan form.
- Now we simply observe that for a block-diagonal matrix, the minimal polynomial of the full matrix is the least common multiple of the minimal polynomial of each block on the diagonal, since each block must individually be annihilated by the minimal polynomial.
- Putting all of this together shows immediately that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ.

(i) Show that a matrix is diagonalizable over $\mathbb{C}$ if and only if its minimal polynomial has no repeated roots.

- A matrix is diagonalizable if and only if all of the blocks in its Jordan form have size 1.

- But by (h), the exponent of $t-\lambda$ in the minimal polynomial is the size of the largest Jordan block.
- Thus, A is diagonalizable if and only if the exponent of $t-\lambda$ is 1, for every eigenvalue λ. Since all eigenvalues are roots of the minimal polynomial by (d), the result follows immediately.

(j) Show that the minimal polynomial of a 2×2 matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 2×2 matrices with minimal polynomials $m(t) = t^2+t$, t^2+1, and $t-3$ over $\mathbb{C}$.

- If $m=(t-\lambda)^2$ then there is a 2×2 Jordan λ-block. If $m=(t-\lambda)(t-\mu)$ then there is a 1×1 block for both λ and μ. And if $m=t-\lambda$ then there are only 1×1 λ-blocks. In each case the Jordan form is determined completely.
- The Jordan forms are $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

(k) Show the minimal and characteristic polynomials of a 3×3 matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 3×3 matrices with $(m(t), p(t))$ equal to (t, t^3), (t^2, t^3), (t^3, t^3), (t^2-t, t^3-t^2), (t^2-t, t^3-2t^2+t).

- If the minimal and characteristic polynomials are given, then all of the eigenvalues are known and the smallest and largest Jordan block sizes are also known. Since the only possible lists of sizes are $\{3\}$, $\{2,1\}$, $\{2\}$, $\{1,1,1\}$, $\{1,1\}$, $\{1\}$, knowing the total number of eigenvalues and the smallest and largest block sizes uniquely determines which sizes appear for each eigenvalue.
- The Jordan forms are $\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$.

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7. [Challenge] The goal of this problem is to characterize when the limit of matrix powers $\lim_{n\to\infty} A^n$ converges.

(a) Suppose $\lim_{n\to\infty} A^n = B$ exists. Show that every column of B lies in the 1-eigenspace of A. [Hint: Why is $AB = B$?]

- If $\lim_{n\to\infty} A^n = B$, multiplying by A yields $\lim_{n\to\infty} A^{n+1} = AB$. But the limit on the left is also B, by shifting the index of the variable n, so $B = AB$.
- If the ith column of B is $\mathbf{v}_i$, then since the ith column of AB is the matrix product of A with the ith column of B, we see that $A\mathbf{v}_i = \mathbf{v}_i$: thus, $\mathbf{v}_i$ is in the 1-eigenspace of A.

Now let J be a $d\times d$ Jordan block matrix with eigenvalue $\lambda \in \mathbb{C}$ and let $N = J - \lambda I_d$ be the matrix with 1s directly above the diagonal and 0s elsewhere.

(b) Show that $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ for each $n \geq 1$.

- Note that $J^n = (\lambda I_d + N)^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{n}N^n$ by the binomial theorem and the fact that $NI_n = I_n N$.
- However, N^d is the zero matrix, which follows by noting that $N(\mathbf{e}_i) = \mathbf{e}_{i+1}$ where $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard basis, and we view $\mathbf{e}_k = \mathbf{0}$ for $k > d$. Then $N^d(\mathbf{e}_i) = \mathbf{e}_{i+d} = \mathbf{0}$ for all i, so N^d is zero on all vectors.
- Thus, the terms past N^d are all zero, so we may ignore them. Thus we get $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ as claimed.

(c) Show that $\lim_{n\to\infty} J^n$ exists if and only if $|\lambda| < 1$ or if $\lambda = 1$ and $d = 1$.

- From (a) we see that $J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \ddots & \binom{n}{d-1}\lambda^{n-d+1} \\ & \lambda^n & \ddots & \ddots \\ & & \ddots & n\lambda^{n-1} \\ & & & \lambda^n \end{bmatrix}$.
- Clearly $J^n = [1]$ if $\lambda = 1$ and $d = 1$ so J^n converges in that case, and if $|\lambda| < 1$ then every entry in J^n converges to zero so J^n converges as well.

- Conversely, in order for $\lim_{n\to\infty} J^n$ to exist, we require λ^n to converge as $n \to \infty$. This clearly requires $|\lambda| \le 1$ since otherwise $|\lambda^n| \to \infty$, and if $|\lambda| = 1$ then if $\lambda = e^{i\theta}$ then $\lambda^n = e^{in\theta}$, which does not converge as $n \to \infty$ unless $\lambda = 1$. Furthermore, if $\lambda = 1$ and $d > 1$, then the entries immediately above the diagonal in J^n are equal to n, which does not converge.
- Therefore, if $\lim_{n\to\infty} J^n$ exists, then we must have $|\lambda| < 1$ or $\lambda = 1$ and $d = 1$, as claimed.

(d) Let A be a square complex matrix. Show that $\lim_{n\to\infty} A^n$ exists if and only if 1 is the only eigenvalue of A of absolute value ≥ 1 and the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial.

- If J is the Jordan canonical form of A with $J = PAP^{-1}$, then $J^n = PA^nP^{-1}$ and $A^n = P^{-1}J^nP$, so $\lim_{n\to\infty} A^n$ exists if and only if $\lim_{n\to\infty} J^n$ exists.
- Since J is block-diagonal, $\lim_{n\to\infty} J^n$ exists if and only if the limit of the nth power of each Jordan block in J exists. But by part (b), this is the case if and only if each Jordan block either has $|\lambda| < 1$, or if $\lambda = 1$ and $d = 1$.
- Equivalently, this means that the only eigenvalue of absolute value ≥ 1 is $\lambda = 1$, and if each Jordan block with $\lambda = 1$ has size 1. This is equivalent to saying that every generalized 1-eigenvector is a 1-eigenvector, which is in turn equivalent to saying that the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial, as claimed.

(e) Suppose M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1) such that some power of M has all positive entries. Show that $\lim_{n\to\infty} M^n$ converges to a matrix whose columns are all 1-eigenvectors of M. [Hint: Use the results of the challenge problem from homework 9 applied to an appropriate power of M.]

- Suppose M^n has all positive entries. Then M^n is still a stochastic matrix, so by the challenge problem from homework 9, we know that all eigenvalues of M^n satisfy $|\lambda| < 1$ or $\lambda = 1$, and the 1-eigenspace has dimension 1.
- By the spectral mapping theorem, if μ is an eigenvalue of M then μ^n is an eigenvalue of M^n, so either $|\mu^n| < 1$ or $\mu^n = 1$, and the total dimension of all the eigenspaces with $\mu^n = 1$ is 1.
- But since 1 is an eigenvalue of M as well, it must be the only eigenvalue with $\mu^n = 1$, and then the remaining eigenvalues have $|\mu| < 1$.
- Hence by (c), the matrix limit $\lim_{n\to\infty} M^n$ converges. More specifically, if J is the Jordan canonical form of M with $M = P^{-1}JP$, where the (1,1)-entry of J is 1 and the first column of P is a 1-eigenvector of M, then $\lim_{n\to\infty} J^n$ is the matrix with (1,1)-entry equal to 1 and other entries equal to zero. It is then straightforward to compute that the product $\lim_{n\to\infty} M^n = P^{-1}[\lim_{n\to\infty} J^n]P$ has all columns proportional to the first column of P, as claimed.
- Finally, by (a), since $\lim_{n\to\infty} M^n$ converges, all of the columns are 1-eigenvectors of M.
