E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2025 ~ Homework 1, due Fri Jan 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:

   (a) If $f : A \to A$ is a one-to-one function, then f must be onto.

   (b) The set of integers $\mathbb{Z}$ is not a field.

   (c) Every field has infinitely many elements.

   (d) It is impossible to have $6 = 0$ in a field F.

   (e) There is a system of linear equations over $\mathbb{R}$ having exactly two different solutions.

   (f) For any $n \times n$ matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$.

   (g) For any $n \times n$ matrices A and B, $(BA)^T = B^T A^T$.

   (h) For any invertible $n \times n$ matrices A and B, $(A + B)^{-1} = A^{-1} + B^{-1}$.

   (i) For any invertible $n \times n$ matrices A and B, $(BA)^{-1} = A^{-1}B^{-1}$.

   (j) If A and B are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.

   (k) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(2A) = 3n$.

   (l) For any $n \times n$ matrix A, $\det(A) = -\det(A^T)$.

   (m) For any $n \times n$ matrices A and B, $\det(AB) = \det(B)\det(A)$.

   (n) If the coefficient matrix of a system of 6 linear equations in 6 unknowns is invertible, then the system has infinitely many solutions.

   (o) If p and q are polynomials in $F[x]$ of the same degree n, then $p + q$ also has degree n.

   (p) If p and q are polynomials in $F[x]$ of the same degree n, then $p \cdot q$ has degree n^2.

2. Find the general solution to each system of linear equations:

   (a) $\left\{\begin{array}{l} -x - 3y + 5z = 9 \\ 3x + 2y + 2z = 0 \\ 2x + 2y + 3z = 4 \end{array}\right\}$.

   (b) $\left\{\begin{array}{l} x - 2y + 4z = 4 \\ 2x + 4y + 8z = 0 \end{array}\right\}$.

   (c) $\left\{\begin{array}{l} a + b + c + d \quad\;\;\, = 2 \\ a + b + c \quad\;\; + e = 3 \\ a + b \quad\; + d + e = 4 \\ a \quad\; + c + d + e = 5 \\ \quad\;\; b + c + d + e = 6 \end{array}\right\}$.

   (d) $\left\{\begin{array}{l} x + 3y + \;\; z = -4 \\ -x - 6y + 8z = 10 \\ 2x + 4y + 8z = 0 \end{array}\right\}$.

   (e) $\left\{\begin{array}{ll} a + b + c + d + e & = 1 \\ a + 2b + 3c + 4d + 5e & = 6 \end{array}\right\}$.

3. Compute the following things:

   (a) If $\mathbf{v} = (3, 0, -4)$ and $\mathbf{w} = (-1, 6, 2)$ in $\mathbb{R}^3$, find $\mathbf{v} + 2\mathbf{w}$, $||\mathbf{v}||$, $||\mathbf{w}||$, $||\mathbf{v} + 2\mathbf{w}||$, and $\mathbf{v} \cdot \mathbf{w}$.

   (b) The sum and product of the polynomials $2x + 3$ and $x^2 - 1$ in $\mathbb{R}[x]$.

   (c) The reduced row-echelon forms of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 2 & 3 \\ 2 & 1 & 0 & -1 & -2 \\ -4 & -2 & 0 & 3 & 0 \end{bmatrix}$.

   (d) The determinants of $\begin{bmatrix} -1 & 5 & 2 \\ 0 & -3 & 7 \\ 2 & 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$.

   (e) The inverses of $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -3 & -2 \\ -3 & 7 & 8 \\ 2 & -6 & -5 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let $\mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$.

   (a) Prove that $||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2 = 2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2$.

   (b) Deduce that in any parallelogram, the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides. [Hint: Suppose the sides are vectors $\mathbf{v}$ and $\mathbf{w}$.]

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5. Suppose that A and B are $n \times n$ matrices with entries from a field F.

   (a) If AB is invertible, show that A and B are invertible.

   (b) If A is invertible, show that A^T is invertible and that its inverse is $(A^{-1})^T$.

   (c) If $I_n + AB$ is invertible, show that $I_n + BA$ is also invertible. [Hint: Consider $M = I_n - B(I_n + AB)^{-1}A$.]

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6. Let F be a field of characteristic not 2 (i.e., in which $2 \neq 0$). A square matrix A with entries from F is called symmetric if $A = A^T$ and skew-symmetric if $A = -A^T$.

   (a) For any $n \times n$ matrix B, show that $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

   (b) Show that any square matrix M can be written *uniquely* in the form $M = S + T$ where S is symmetric and T is skew-symmetric. [Make sure to prove that there is *only* one such decomposition!]

   (c) If A is a skew-symmetric $n \times n$ real matrix and n is odd, show that $\det(A) = 0$.

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7. Prove the following things via induction (or otherwise):

   (a) The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. (Thus $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so forth.) Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for every positive integer n.

   (b) Prove that the nth power of the matrix $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 - 2n & 4n \\ -n & 1 + 2n \end{bmatrix}$ for each positive integer n.

   (c) Let M_n be the $n \times n$ matrix with 1s on the diagonal and directly below the diagonal, -1s directly above the diagonal, and 0s elsewhere. Prove that $\det(M_n)$ is the $(n+1)$st Fibonacci number F_{n+1}.

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8. [Challenge] Let F be a field and suppose $x_1, \ldots, x_n$ are elements of F. The goal of this problem is to evaluate the famous Vandermonde determinant $V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$.

   (a) Show that if any of the x_i are equal to one another, then $V(x_1, \ldots, x_n) = 0$.

   (b) Show that as a polynomial in the variables $x_1, \ldots, x_n$, $V(x_1, \ldots, x_n)$ has degree $\frac{n(n-1)}{2}$ and is divisible by $x_j - x_i$ for any $i \neq j$. [Hint: Use (a) and the remainder theorem.]

   (c) Deduce that $V(x_1, \ldots, x_n)$ is divisible by the product $\prod_{1 \leq i < j \leq n}(x_j - x_i)$ and that this product is a polynomial of degree $\frac{n(n-1)}{2}$.

   (d) Show in fact that $V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$. [Hint: Compare degrees and coefficients of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.]

   (e) Suppose that $x_1, \ldots, x_n \in F$ are distinct and $y_1, \ldots, y_n \in F$ are arbitrary. Prove that there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ in $F[x]$ of degree at most $n - 1$ such that $p(x_i) = y_i$ for each $1 \leq i \leq n$. [Hint: Write down the corresponding system of linear equations.]