

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Determine, with brief reasons, whether each subset S is an ideal of the given ring R :
 - (a) $R = F[x]$, $S =$ the set of polynomials whose coefficient of x is zero.
 - (b) $R = \mathbb{Z}/18\mathbb{Z}$, $S = \{0, 3, 6, 9, 12, 15\}$.
 - (c) $R = \mathbb{Z}/15\mathbb{Z}$, $S = \{0, 4, 8, 12\}$.
 - (d) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(a, a) : a \in \mathbb{Z}\}$.
 - (e) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(0, a) : a \in \mathbb{Z}\}$.
 - (f) $R = F[x]$, $S = F[x^2]$, the polynomials in which only even powers of x appear.
 - (g) $R = F[x]$, $S =$ the set of polynomials whose coefficients sum to zero.

 2. Let $R = \mathbb{Z}[\sqrt{7}]$ and consider the ideals $I = (3)$ and $J = (3, 1 + \sqrt{7})$.
 - (a) Show R/I has exactly 9 residue classes. [Hint: They are $p + q\sqrt{7} + I$ for $p, q \in \{0, 1, 2\}$. Explain why.]
 - (b) Write down the multiplication table for R/I , and identify which elements are units and which elements are zero divisors. Is I a prime ideal? A maximal ideal?
 - (c) Show that R/J contains exactly 3 residue classes and identify them. Is J a prime ideal? A maximal ideal?
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Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Let R be a commutative ring with 1.
 - (a) If S is a not necessarily finite set of ideals of R , show that the intersection $T = \bigcap_{I \in S} I$ is an ideal of R .
 - (b) Show via an explicit example that the union of a collection of ideals of R is not necessarily an ideal of R .
 - (c) If $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$ is a not necessarily finite increasing chain of ideals of R , show that the union $T = \bigcup_{i=1}^{\infty} I_i$ is an ideal of R .

 4. Let R be a commutative ring with 1 and define the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$. Prove the binomial theorem in R : $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ for any $x, y \in R$ and any $n > 0$.

 5. Let R be a commutative ring with 1 and let I and J be ideals of R .
 - (a) Show that $I + J = \{a + b : a \in I, b \in J\}$, the set of all sums of elements of I and J , is an ideal of R .
 - (b) Show that $I + J$ is the smallest ideal of R that contains both I and J . Deduce that if $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$ then $I + J = (a_1, \dots, a_n, b_1, \dots, b_m)$.
 - (c) Let a and b be positive integers with greatest common divisor d . Show that $(a) + (b) = (d)$ in \mathbb{Z} .
 - (d) Show that $IJ = \{a_1 b_1 + \dots + a_n b_n, : a_i \in I, b_i \in J\}$, the set of finite sums of products of an element of I with an element of J , is an ideal of R .
 - (e) If $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$, show that $IJ = (a_1 b_1, a_1 b_2, \dots, a_n b_1, a_n b_2, \dots, a_n b_m)$.
 - (f) Show that IJ is an ideal contained in $I \cap J$, and give an example where $IJ \neq I \cap J$.
 - (g) If $I + J = R$, show that $IJ = I \cap J$. [Hint: There exist $x \in I$ and $y \in J$ with $x + y = 1$.]
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6. Suppose R is a finite ring with $1 \neq 0$. If R has a prime number of elements p , show that R is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ as a ring. [Hint: Use Lagrange's theorem on the additive group of R to show 1 has additive order p , then consider the map $\varphi: \mathbb{Z} \rightarrow R$ with $\varphi(n) = n1_R$.]

7. Let F be a field and define $R = F[\epsilon]/(\epsilon^2)$, a ring known as the ring of dual numbers over F . Intuitively, one can think of the element $\epsilon \in R$ as being like an “infinitesimal”: a number so small that its square is zero.

- (a) Show that the zero divisors in R are the elements of the form $b\epsilon$ with $b \neq 0$, and the units in R are the elements of the form $a + b\epsilon$ with $a \neq 0$.
- (b) Find the three ideals of R .
- (c) Let $S = R[x]$ and $p(x) \in S$. Show that $p(x + \epsilon) = p(x) + \epsilon p'(x)$ in S , where $p'(x)$ denotes the derivative of $p(x)$.
- (d) Let $p(x), q(x) \in F[x]$ and set $P(x) = p(x)q(x)$. Show that $P'(x) = p'(x)q(x) + p(x)q'(x)$. [Hint: Use (c).]

Remark: Part (c) shows how to use dual numbers to give a purely algebraic way to compute the derivative of a polynomial (in fact, some computer systems actually do differentiation this way), and (d) illustrates that they yield a formal proof of the product rule. In fact, the dual numbers are essentially the same object used in the construction of cotangent spaces in differential geometry.

8. [Challenge] The goal of this problem is to prove the following theorem of Hurwitz: if α has a continued fraction $\alpha = [a_0, a_1, \dots]$ with convergents $p_n/q_n = [a_0, \dots, a_n]$ and remainders $\alpha_n = [a_{n+1}, \dots]$, then for any k at least one of the inequalities $|\alpha - p_{n-2}/q_{n-2}| < \frac{1}{q_{n-2}^2\sqrt{5}}$, $|\alpha - p_{n-1}/q_{n-1}| < \frac{1}{q_{n-1}^2\sqrt{5}}$, $|\alpha - p_n/q_n| < \frac{1}{q_n^2\sqrt{5}}$ must hold.

For each n , define $\varphi_n = q_{n-2}/q_{n-1}$ and also set $\psi_n = \varphi_n + \alpha_n$.

- (a) Show that $\left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{q_n^2 \psi_{n+1}}$. [Hint: Use $\alpha = \frac{p_n \alpha_{n+1} + p_{n-1}}{q_n \alpha_{n+1} + q_{n-1}}$.]
 - (b) Show that $\frac{1}{\varphi_{n+1}} + \frac{1}{\alpha_{n+1}} = \psi_n$.
 - (c) Show that if $\psi_n, \psi_{n-1} \leq \sqrt{5}$ then $\varphi_n > \frac{\sqrt{5}-1}{2}$. [Hint: Use $\varphi_n + \alpha_n \leq \sqrt{5}$ and $\frac{1}{\varphi_n} + \frac{1}{\alpha_n} \leq \sqrt{5}$ to bound φ_n , noting that φ_n is rational.]
 - (d) Show that $a_n = \frac{1}{\varphi_{n+1}} - \varphi_n$.
 - (e) Show that at least one of the inequalities $|\alpha - p_{n-2}/q_{n-2}| < \frac{1}{q_{n-2}^2\sqrt{5}}$, $|\alpha - p_{n-1}/q_{n-1}| < \frac{1}{q_{n-1}^2\sqrt{5}}$, $|\alpha - p_n/q_n| < \frac{1}{q_n^2\sqrt{5}}$ must hold. [Hint: Apply (a), then (c), and finally estimate a_n using (d).]
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