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1. Find all ordered pairs (a, b) of positive integers for which $\frac{1}{a} + \frac{2}{b} = \frac{1}{10}$.

   - Clearing the denominators yields $10b + 20a = ab$, and rearranging and factoring then yields $(a-10)(b-20) = 200$. Since a, b are positive we see that $a - 10, b - 20$ must in fact be positive since otherwise the product $(a-10)(b-20)$ is less than 200.
   - For each integer factorization of 200 (namely, $1 \cdot 200$, $2 \cdot 100$, $4 \cdot 50$, $5 \cdot 40$, $8 \cdot 25$, $10 \cdot 20$, $20 \cdot 10$, $25 \cdot 8$, $40 \cdot 5$, $50 \cdot 4$, $100 \cdot 2$, $200 \cdot 1$) we obtain a solution (a, b) to the system: $\boxed{(11, 220),\ (12, 120),\ (14, 70),\ (15, 60),\ (18, 45),\ (20, 40),\ (30, 30),\ (35, 28),\ (50, 25),\ (60, 24),\ (110, 22),\ (210, 21)}$.

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2. Find all ordered pairs (a, b) of positive integers for which $\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$.

   - Clearing denominators yields $2018(a + b) = 3ab$, and so multiplying by 3, rearranging, and factoring yields $(3a - 2018)(3b - 2018) = 2018^2 = 2^2 \cdot 1009^2$ (note 1009 is prime).
   - Since $3a - 2018$ and $3b - 2018$ are both greater than -2018, we see that both must be positive, and so $3a - 2018$ must be a factor of $2^2 \cdot 1009^2$ congruent to $-2018 \equiv 1 \bmod 3$. This yields possibilities of $3a - 2018 = 1, 4, 1009, 4 \cdot 1009, 1009^2, 4 \cdot 1009^2$ so that $(a, b) = \boxed{(673, 1358114),\ (674, 340033),\ (1009, 2018),\ (2018, 1009),\ (340033, 674),\ (1358114, 673)}$.

   Remark: This is problem A1 from the 2018 Putnam exam.

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3. Find all solutions to the Diophantine equation $y^2 = x^4 + 2x^3 + 2x^2 + 4$.

   - Let $p(x) = x^4 + 2x^3 + 2x^2 + 4$. Note that $p(x) - (x^2 + x)^2 = x^2 + 4 > 0$, and also $p(x) - (x^2 + x + 1)^2 = 3 - 2x - x^2$, so unless $3 - 2x - x^2 \le 0$, which is to say, whenever $-3 \le x \le 1$, we would have $(x^2 + x)^2 < p(x) < (x^2 + x + 1)^2$, which would be impossible since then we would have $x^2 + x < y < x^2 + x + 1$.
   - Thus we must have $-3 \le x \le 1$. Testing these five values produces the solutions $(x, y) = \boxed{(-3, \pm 7),\ (0, \pm 2),\ (1, \pm 3)}$.

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4. Find all integers n such that $n^4 + n^3 + n^2 + n + 1$ is a perfect square.

   - Let $p(n) = n^4 + n^3 + n^2 + n + 1$. We bound $16p(n)$ above and below by perfect squares.
   - First, $16p(n) - (4n^2 + 2n + 1)^2 = 4n^2 + 12n + 15 = (2n + 3)^2 + 6$, so $16p(n) > (4n^2 + 2n + 1)^2$.
   - Also, $16p(n) - (4n^2 + 2n + 2)^2 = -4n^2 + 8n + 12 = 16 - (2n - 2)^2$, so unless $16 \ge (2n - 2)^2$ which is to say $-1 \le n \le 3$, then we have $16p(n) < (4n^2 + 2n + 2)^2$.
   - Thus, unless $-1 \le n \le 3$, it is true that $(4n^2 + 2n + 1)^2 < p(n) < (4n^2 + 2n + 2)^2$ and so $p(n)$ cannot be the square of an integer. Thus, the only possible n have $-1 \le n \le 3$. Testing $p(n)$ for $-1 \le n \le 3$ shows that the only values of n for which $p(n)$ are a perfect cube are $n = \boxed{-1,\ 0,\ 3}$.

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5. Let $n \ge 2$ be a fixed integer. Find infinitely many distinct positive integer triples (x, y, z) such that $x^n + y^n = z^{n+1}$. [Hint: Divide both sides by z^n.]

   - Following the hint, dividing both sides by z^n yields $(x/z)^n + (y/z)^n = z$.
   - We want to arrange for x, y, z to be integers, so a natural way is just to require x/z to be some integer a and y/z to be some integer b.
   - Then $z = a^n + b^n$ and then $x = a(a^n + b^n)$ with $y = b(a^n + b^n)$, and since we can choose a, b arbitrarily we get infinitely many triples.

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6. The goal of this problem is to solve the equation $x^y = y^x$ in positive rational numbers. Assume $x, y > 0$.

   (a) Prove that any rational solution with $y > x$ is of the form $(x, y) = ((1 + 1/u)^u, (1 + 1/u)^{u+1})$ for some rational number $u > 0$. [Hint: If $y > x$, set $y = (1 + 1/u)x$.]

   - If $x = 1$ then clearly $y = 1$. Now assume $x \neq 1$ and $y \neq x$. Without loss of generality we can also take $y > x$: then $y/x - 1$ is a positive rational number, so we can set $y/x = 1 + 1/u$ for some $u > 0$.
   - Plugging in $y = (1 + 1/u)x$ then yields $x^{(1+1/u)x} = [(1 + 1/u)x]^x$.
   - Now taking the xth root of both sides and dividing by x yields $x^{1/u} = 1+1/u$, so that $x = (1+1/u)^u$. Then $y = (1 + 1/u)x = (1 + 1/u)^{u+1}$, so (x, y) is as claimed.

   (b) Let $m \geq 2$. Show that the difference between any two positive consecutive mth powers is greater than m.

   - This follows from the binomial theorem: for $m \geq 2$ we have $(a+1)^m - a^m = ma^{m-1} + \binom{m}{2}a^{m-2} + \cdots + 1$ and so for $m \geq 2$ the terms ma^{m-1} and 1 are distinct, and their sum is at least $m \cdot 1^{m-1} + 1 = m + 1$.
   - It is also possible to establish the result by induction on a or on m, or by noting that $(a + 1)^m - a^m$ is increasing as a function of a, so it is at least $2^m - 1 > m$.

   (c) With notation as in part (a), suppose $u = n/m$ in lowest terms. Show that $m + n$ and n must both be mth powers and deduce that $m = 1$. [Hint: Write out x in terms of m, n and use the fact that $m+n, m, n$ are relatively prime.]

   - Using (a), we see that if $u = n/m$ then $x = (1 + 1/u)^u = (m + n)^{n/m}/n^{n/m}$.
   - Then because $m + n$ and n are relatively prime, the expression $(m + n)^{n/m}/n^{n/m}$ is rational only when both $(m + n)^{n/m}$ and $n^{n/m}$ are rational numbers, and since m, n are relatively prime, this occurs only when $m + n$ and n are mth powers.
   - But by part (b), it cannot be the case that $m + n$ and n are both mth powers if $m \geq 2$, since their difference is only m.
   - Thus, we must have $m = 1$ as claimed.

   (d) Conclude that the rational solutions to $x^y = y^x$ are of the form $(x, y) = (s, s)$ for rational s along with $(x, y) = ((1 + 1/n)^n, (1 + 1/n)^{n+1})$ or $((1 + 1/n)^{n+1}, (1 + 1/n)^n)$ for integers n.

   - Clearly if $y = x$ then $x^y = y^x$ so all (s, s) with s rational are solutions. If $y > x$ then by (a) and (c) then $(x, y) = ((1 + 1/n)^n, (1 + 1/n)^{n+1})$, and if $y < x$ then swapping x, y also yields a solution, so we must instead have $(x, y) = ((1 + 1/n)^{n+1}, (1 + 1/n)^n)$.

   (e) Find all integral solutions to $x^y = y^x$.

   - We simply have to determine the possible integer results of the expressions in (d).
   - Clearly $(x, y) = (s, s)$ works for any positive integer s.
   - Also, if n is an integer, then for $(1 + 1/n)^n$ to be an integer we must have $n = 1$, otherwise we have a denominator $n^n > 1$, so the only other integral solutions are $(2, 4)$ and $(4, 2)$.

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7. Prove that the sum of the first n positive integers is a perfect square for infinitely many values of n, and find the first five such n.

- We have the well-known formula $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ as is easy to prove by induction.
- We are therefore seeking solutions to the Diophantine equation $\dfrac{n(n+1)}{2} = k^2$. Multiplying by 8 and completing the square on the left yields $(2n + 1)^2 - 1 = 8k^2$, so that $(2n + 1)^2 - 8k^2 = 1$.
- This is a Pell equation $x^2 - 8y^2 = 1$: since x must always be odd since $8y^2$ is even, each solution of this Pell equation yields an admissible value for n.
- From our study of this Pell equation we know it has infinitely many solutions given by computing powers of the fundamental unit that we can easily calculate as $u = 3 + \sqrt{8}$.
- The first five solutions correspond to $u = 3 + \sqrt{8}$, $u^2 = 17 + 12\sqrt{2}$, $u^3 = 99 + 70\sqrt{2}$, $u^4 = 577 + 408\sqrt{2}$, and $u^5 = 3363 + 2378\sqrt{2}$, which give $n = \boxed{1, 8, 49, 288, 1681}$.

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8. Prove that there are no integral solutions to the equation $x^2 + y^2 = 3z^2$ other than $(0, 0, 0)$. [Hint: Use a descent argument modulo 3.]

   - Clearly, $(0, 0, 0)$ is a solution. Now suppose there is another solution: without loss of generality suppose z is positive and minimal.
   - Reducing both sides modulo 3 yields $x^2 + y^2 \equiv 0 \pmod 3$. Since squares are 0 or 1 mod 3, both x and y must be divisible by 3: say $x = 3x'$ and $y = 3y'$.
   - Then $3(x'^2 + y'^2) = z^2$ so z is also divisible by 3, say with $z = 3z'$. But now we have $(x')^2 + (y')^2 = 3(z')^2$ and $z' < z$, which is impossible since we assumed z was minimal. Thus, there are no solutions.

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9. Find all positive integers n such that there exist positive integers a, b, c with $2a^n + 3b^n = 4c^n$. [Hint: Do $n = 2$ and $n > 2$ separately, and use descent arguments.]

   - The answer is only $n = 1$. Clearly $n = 1$ works since for instance we can take $a = b = 4$ and $c = 5$. Now assume $n \geq 2$ and that $2a^n + 3b^n = 4c^n$ where $a + b + c$ is minimal.
   - For $n = 2$, reducing both sides modulo 3 yields $2a^2 \equiv c^2 \pmod 3$. Since 2 is not a square modulo 3, if a or c is not divisible by 3 we would obtain an immediate contradiction. Hence both a and c must be divisible by 3, but then $3b^2$ would be divisible by 9 hence $3|b$ also. This cannot occur since we could then divide a, b, c by 3 to get a smaller triple.
   - For $n \geq 3$, since $3b^n = 4c^n - 2a^n$ we see that $3b^n$ is even hence b is even. Then $2a^n = 4c^n - 3b^n$ is divisible by 4, whence a^n is even hence a is even. Then finally $4c^n = 2a^n + 3b^n$ is divisible by $2^n \geq 8$ hence c is also even. This cannot occur since we could then divide a, b, c by 2 to get a smaller triple.

   Remark: This is problem A1 from the 2024 Putnam exam.

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10. The goal of this problem is to find all solutions to the Diophantine equation $x^2 + y^2 = 2z^2$ in various ways. If $\gcd(x, y, z) = d$ then clearly $(x/d, y/d, z/d)$ is also a solution, so it suffices to find all primitive solutions, in which $\gcd(x, y, z) = 1$. So now suppose that (x, y, z) is a primitive solution to $x^2 + y^2 = 2z^2$.

    (a) Show that x and y must have the same parity. Letting $a = (x - y)/2$ and $b = (x + y)/2$, show that there exist integers s, t such that a and b equal $2st$ and $s^2 - t^2$ in some order.

    - Note that if x, y have opposite parity then $x^2 + y^2$ is odd and thus cannot equal $2z^2$. So x, y have the same parity, meaning that $a = (x - y)/2$ and $b = (x + y)/2$ are both nonnegative integers. If $x \not\equiv y \pmod 4$ then replacing x with $-x$ makes $x \equiv y \pmod 4$ in which case a is even and b is odd.
    - Then $a^2 + b^2 = (x^2 + y^2)/2$, so the original equation is equivalent to $a^2 + b^2 = z^2$. This is the original Pythagorean equation, so since a is even we get $a = 2st$, $b = s^2 - t^2$, $z = s^2 + t^2$ as required.

    (b) In the ring of Gaussian integers $\mathbb{Z}[i]$, show that $1 + i$ must divide both $x + iy$ and $x - iy$. Letting $p + iq = (x + iy)/(1 + i)$, show that there exist integers s, t such that p and q equal $s^2 - t^2$ and $2st$ in some order.

    - Factoring in $\mathbb{Z}[i]$ yields $(x + iy)(x - iy) = -i(1 + i)^2 z^2$.
    - Since $1 + i$ is a Gaussian prime it must divide one of $x + iy$ and $x - iy$, but if it divides one then it divides the other since they differ by $2iy$, a multiple of $1 + i$.
    - If we set $x + iy = (1 + i)(p + iq)$ then we are reduced to $(p + iq)(p - iq) = z^2$, which per our analysis has solution $p = s^2 - t^2$ and $q = 2st$, possibly after scaling by i (which would interchange p and q).

    (c) Show that the line through $(x/z, y/z)$ and $(-1, 1)$ has rational slope. Also, if ℓ is the line with rational slope t/s through the point $(-1, 1)$, find the intersection point of ℓ with the circle $(x/z)^2 + (y/z)^2 = 2$.

    - Note that the line through two points with rational coordinates necessarily have rational slope.
    - Taking lines with rational slope t/s through the point $(-1, 1)$ on the circle $x^2 + y^2 = 2$ yields the other intersection point as $(x/z, y/z) = \left(\dfrac{s^2 - 2st - t^2}{s^2 + t^2}, \dfrac{s^2 + 2st - t^2}{s^2 + t^2} \right)$.

    (d) Find all primitive solutions to the Diophantine equation $x^2 + y^2 = 2z^2$.

    - Using any of the three results yields $(x, y, z) = \boxed{(s^2 + 2st - t^2, s^2 - 2st - t^2, s^2 + t^2)}$ for $s, t \in \mathbb{Z}$.

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11. Prove that there are no integral solutions to the equation $y^9 = x^2 + 2024^{2020}$. [Hint: Work modulo 19.]

- Modulo 19, we have $(y^9)^2 \equiv y^{18} \equiv 1 \pmod{19}$ if y is not divisible by 19 by Euler's theorem, and so $y^9 \in \{-1, 0, 1\}$ mod 19.
- Likewise, we can simply list all of the squares modulo 19: they are $\{0, 1, 4, 5, 6, 7, 9, 11, 16, 17\}$.
- Therefore, $y^9 - x^2 \in \{0, 1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ mod 19.
- However, we can compute $2024^{2020} \equiv 10^{2020} \equiv 10^4 \equiv 100^2 \equiv 5^2 \equiv 6 \pmod{19}$. Since this is not one of the possible residue classes of the form $y^9 - x^2$ by the above calculation, we deduce that there are no solutions to $y^9 = x^2 + 2024^{2020}$ mod 19, as claimed.

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12. [Challenge] The goal of this problem is to give two ways to solve the Diophantine equation $(x^2 - xy - y^2)^2 = 1$ in positive integers. Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, and let $\varphi = \frac{1+\sqrt{5}}{2}$.

(a) Show that $(x, y) = (F_{n+1}, F_n)$ satisfies $(x^2 - xy - y^2)^2 = 1$ for every $n \geq 1$.

- Induction on n. The result clearly holds for $n = 1$ since $(1^2 - 1 - 1)^2 = 1$.
- For the inductive step, suppose $(F_{n+1}^2 - F_{n+1}F_n - F_n^2)^2 = 1$.
- Then $F_{n+2}^2 - F_{n+2}F_{n+1} - F_{n+1}^2 = (F_{n+1} + F_n)^2 - (F_{n+1} + F_n)F_{n+1} - F_{n+1}^2 = -(F_{n+1}^2 - F_{n+1}F_n - F_n^2)$, so its square is also 1 by the inductive hypothesis.

We now show that the pairs of consecutive Fibonacci numbers are the only solutions. The first approach is via a descent argument.

(b) Suppose $(x, y) = (a, b)$ is a solution to $(x^2 - xy - y^2)^2 = 1$. Show that $a \geq b$, and that if $a > b$ then $(x, y) = (b, a - b)$ is also a solution to the system.

- First, we have $a^2 - ab - b^2 = \pm 1$ and so $a^2 \geq b^2 + (ab - 1) \geq b^2$ since $ab \geq 1$. Thus, $a \geq b$.
- Furthermore, if $a > b$ then $b^2 - b(a - b) - (a - b)^2 = -(a^2 - ab - b^2)$, and so if (a, b) is a solution then so is $(b, a - b)$.

(c) Prove that every solution to $(x^2 - xy - y^2)^2 = 1$ is of the form $(a, b) = (F_{n+1}, F_n)$ for some $n \geq 1$.

- The point here is that applying the map $(a, b) \mapsto (b, a - b)$ will always yield a smaller solution, and this process can only terminate in a pair with equal terms.
- But if $y = x$ then $1 = (x^2 - xy - y^2)^2 = x^4$ implies $(x, y) = (1, 1) = (F_1, F_2)$.
- The inverse of this map is $(c, d) \mapsto (c + d, c)$, which clearly maps $(F_{n+1}, F_n) \mapsto (F_{n+2}, F_{n+1})$, and thus by an easy induction, we see that every solution to the system is of the form $(a, b) = (F_{n+1}, F_n)$ for some $n \geq 1$.

Now we give a second approach based on rational approximations.

(d) Suppose that (x, y) is a solution to $\left|x^2 - xy - y^2\right| = 1$ with $x \geq y > 1$. Show that $\left|\frac{x}{y} - \varphi\right| < \frac{1}{2y^2}$. [Hint: Let $t = \frac{x}{y} - \frac{1}{2}$ and then show that $t > \frac{\sqrt{5}}{2} + \frac{1}{2y^2}$ and $t < \frac{\sqrt{5}}{2} - \frac{1}{2y^2}$ both yield contradictions.]

- Let $t = \frac{x}{y} - \frac{1}{2}$ so that $t^2 = \frac{x^2}{y^2} - \frac{x}{y} + \frac{1}{4}$, and then observe that $\left|x^2 - xy - y^2\right| = y^2 \cdot |t^2 - \frac{5}{4}|$. Thus, we have $|t^2 - \frac{5}{4}| = \frac{1}{y^2}$, so $t^2 = \frac{5}{4} \pm \frac{1}{y^2}$.
- If $t > \frac{\sqrt{5}}{2} + \frac{1}{2y^2}$ then we would have $t^2 > \frac{5}{2} + \frac{\sqrt{5}}{2y^2} + \frac{1}{4y^4} > \frac{5}{4} + \frac{1}{y^2}$, contradiction. Likewise, if $t < \frac{\sqrt{5}}{2} - \frac{1}{2y^2}$ then $t^2 < \frac{5}{2} - \frac{\sqrt{5}}{2y^2} + \frac{1}{4y^4} \leq \frac{5}{4} - \frac{1}{y^2}$ for $y \geq 2$, since in that case we have $\frac{1}{4y^4} \leq [\frac{\sqrt{5}}{2} - 1]\frac{1}{y^2}$. Thus, we must have $\left|\frac{x}{y} - \varphi\right| = \left|t - \frac{\sqrt{5}}{2}\right| < \frac{1}{2y^2}$, as claimed.

(e) Deduce that every solution to the system is of the form $(x, y) = (F_{n+1}, F_n)$.

- This is a direct check for $y = 1$. If $y > 1$, then (d) implies that any solution to $\left|x^2 - xy - y^2\right| = 1$ then $|x/y - \varphi| < \frac{1}{2y^2}$. By our results on continued fractions, this means x/y is a continued fraction convergent to $\varphi = [\overline{1}]$, but as we noted in class, the continued fraction convergents of $[\overline{1}]$ are precisely the ratios F_{n+1}/F_n of consecutive Fibonacci numbers.
- Hence $x/y = F_{n+1}/F_n$. Since clearly x, y are relatively prime this forces $(x, y) = (F_{n+1}, F_n)$. Since all of these pairs are solutions as noted in (a), they are all of the solutions.
