1. For each value of D, use the super magic box to (i) find the continued fraction expansion for $\sqrt{D}$, (ii) find the fundamental unit in the ring $\mathbb{Z}[\sqrt{D}]$, (iii) determine whether the Pell's equation $x^2 - Dy^2 = -1$ has a solution and if so find the smallest one, and (iv) find the smallest two solutions to the Pell's equation $x^2 - Dy^2 = 1$:

(a) $D = 19$.

- Here is the result of doing the super magic box calculation for $D = 19$:

n	-1	0	1	2	3	4	5	6
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	4	2	3	3	2	4
$C_n = (D - A_n^2)/C_{n-1}$		1	3	5	2	5	3	$\boxed{1}$
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		4	2	1	3	1	2	8
$p_n = a_n p_{n-1} + p_{n-2}$	1	4	9	13	48	61	170	1421
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	2	3	11	14	39	326
$p_n^2 - Dq_n^2$		-3	5	-2	5	-3	1	-3

- The continued fraction expansion is $\sqrt{19} = \boxed{[4, \overline{2, 1, 3, 1, 2, 8}]}$ and the fundamental unit is $\boxed{170 + 39\sqrt{19}}$.
- Since the first C_n equal to ± 1 is equal to 1 for $n = 6$, so there is $\boxed{\text{no solution}}$ to $x^2 - 19y^2 = -1$.
- The smallest solution to $x^2 - 19y^2 = 1$ is $(x, y) = (170, 39)$, and then the next smallest solution is given by the square of the fundamental unit $(170 + 39\sqrt{19})^2 = 57799 + 13260\sqrt{19}$. So the smallest two solutions are $\boxed{(170, 39),\ (57799, 13260)}$.

(b) $D = 22$.

- Here is the result of doing the super magic box calculation for $D = 22$:

n	-1	0	1	2	3	4	5	6
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	4	2	4	4	2	4
$C_n = (D - A_n^2)/C_{n-1}$		1	6	3	2	3	6	$\boxed{1}$
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		4	1	2	4	2	1	8
$p_n = a_n p_{n-1} + p_{n-2}$	1	4	5	14	61	136	197	1712
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	1	3	13	29	42	365
$p_n^2 - Dq_n^2$		-6	3	-2	3	-6	1	-5

- The continued fraction expansion is $\sqrt{22} = \boxed{[4, \overline{1, 2, 4, 1, 2, 8}]}$ and the fundamental unit is $\boxed{197 + 42\sqrt{22}}$.
- We see that the first C_n equal to ± 1 is equal to 1 for $n = 6$, so there is $\boxed{\text{no solution}}$ to $x^2 - 22y^2 = -1$.
- The smallest solution to $x^2 - 22y^2 = 1$ is $(x, y) = (197, 42)$, and the next smallest solution is then given by the square of the fundamental unit $(197 + 42\sqrt{22})^2 = 77617 + 16548\sqrt{22}$. So the smallest two solutions are $\boxed{(197, 42),\ (77617, 16548)}$.

(c) $D = 130$.

- Here is the result of doing the super magic box calculation for $D = 130$:

n	-1	0	1	2	3
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	11	7	11
$C_n = (D - A_n^2)/C_{n-1}$		1	9	9	$\boxed{1}$
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		11	2	2	22
$p_n = a_n p_{n-1} + p_{n-2}$	1	11	23	57	1277
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	2	5	112
$p_n^2 - Dq_n^2$		-9	9	-1	9

- The continued fraction expansion is $\sqrt{130} = \boxed{[11, \overline{2, 2, 22}]}$ and the fundamental unit is $\boxed{57 + 5\sqrt{130}}$.
- We see that the first C_n equal to ± 1 is equal to 1 for $n = 3$, so there is a solution to $x^2 - 130y^2 = -1$ and the smallest one is $\boxed{(57, 5)}$.

- The minimal solution to $x^2 - 130y^2 = 1$ is the square of the fundamental unit $(57+5\sqrt{130})^2 = 6499 + 570\sqrt{130}$. The next solution is the fourth power, which is $(57+5\sqrt{130})^4 = 84474001 + 7408860\sqrt{130}$. So the smallest are $\boxed{(6499, 570),\ (84474001, 7408860)}$.

(d) $D = 61$.

- Here is the result of doing the super magic box calculation for $D = 61$:

n	-1	0	1	2	3	4	5	6	7	8	9	10	11
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	7	5	7	5	4	6	4	5	7	5	7
$C_n = (D - A_n^2)/C_{n-1}$		1	12	3	4	9	5	5	9	4	3	12	$\boxed{1}$
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		7	1	4	3	1	2	2	1	3	4	1	14
$p_n = a_n p_{n-1} + p_{n-2}$	1	7	8	39	125	164	453	1070	1523	5639	24079	29718	440131
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	1	5	16	21	58	137	195	722	3083	3805	56353
$p_n^2 - Dq_n^2$		-12	3	-4	9	-5	5	-9	4	-3	12	-1	12

- The continued fraction expansion is $\sqrt{61} = \boxed{[7, \overline{1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14}]}$ and the fundamental unit is $\boxed{29718 + 3805\sqrt{61}}$.
- We see that the first C_n equal to ± 1 is equal to 1 for $n = 11$, so there is a solution to $x^2 - 61y^2 = -1$ and the smallest one is $\boxed{(29718, 3805)}$.
- The smallest solution will be to $x^2 - 61y^2 = 1$ and it is given by the square of the fundamental unit $(29718 + 3805\sqrt{61})^2 = 1766319049 + 226153980\sqrt{61}$. The next solution is the square of this one, which is the quite lengthy $(29718 + 3805\sqrt{61})^4 = 6239765965720528801 + 798920165762330040\sqrt{61}$. So the smallest two are $\boxed{(1766319049, 226153980),\ (6239765965720528801, 798920165762330040)}$.

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2. Use the super magic box to factor each of the following integers:

(a) 437.

- Here is the result of doing the super magic box calculation for $D = 437$ until we find a perfect square for C_n with n even:

n	-1	0	1	2
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	20	17
$C_n = (D - A_n^2)/C_{n-1}$		1	37	4
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		20	1	9
$p_n = a_n p_{n-1} + p_{n-2}$	1	20	21	
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	1	
$p_n^2 - Dq_n^2$		-37	4	

- We see that for $n = 2$, $C_n = 4$. Thus, from the column for $n = 1$, we see that $p_1^2 = 21^2 \equiv 2^2 \mod 437$.
- Then we compute $\gcd(21 + 2, 437) = 23$ and so we get a factorization $437 = 23 \cdot 19$.

(b) 8137.

- Here is the result of doing the super magic box calculation for $D = 8137$ until we find a perfect square for C_n with n even:

n	-1	0	1	2	3	4	5	6
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	90	58	71	73	44	62
$C_n = (D - A_n^2)/C_{n-1}$		1	37	129	24	117	53	81
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		90	4	1	6	1	2	
$p_n = a_n p_{n-1} + p_{n-2}$	1	90	361	451	3067	3518	10103	
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	4	5	34	39	112	
$p_n^2 - Dq_n^2$		-37	129	-24	117	-53	81	

- We see that for $n = 6$, $C_n = 81$. Thus, from the column for $n = 5$, we see that $p_5^2 = 10103^2 \equiv 9^2 \mod 8137$.
- Then we compute $\gcd(10103 + 3, 8137) = 79$ and so we get a factorization $8137 = 79 \cdot 103$.

(c) 15403.

- Here is the result of doing the super magic box calculation for $D = 15403$ until we find a perfect square for C_n with n even:

n	-1	0	1	2	3	4	5	6
$A_n = a_{n-1}C_{n-1} - A_{n-1}$		0	124	119	111	90	19	119
$C_n = (D - A_n^2)/C_{n-1}$		1	27	46	67	109	138	9
$a_n = \lfloor (A_n + a_0)/C_n \rfloor$		124	9	5	3	1	1	27
$p_n = a_n p_{n-1} + p_{n-2}$	1	124	1117	5709	18244	23953	42197	
$q_n = a_n q_{n-1} + q_{n-2}$	0	1	9	46	147	193	340	
$p_n^2 - Dq_n^2$		-27	46	-67	109	-138	9	

- We see that for $n = 6$, $C_n = 9$. Thus, from the column for $n = 5$, we see that $p_1^2 = 42197^2 \equiv 3^2 \bmod 15403$.
- Then we compute $\gcd(42197 + 3, 15403) = 211$ and so we get a factorization $15403 = 73 \cdot 211$.

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3. In class, we showed how to compute solutions to Pell's equation $x^2 - Dy^2 = \pm 1$ by taking powers of the fundamental solution $u = x_1 + y_1\sqrt{D}$ and then extracting coefficients of the resulting power. The goal of this problem is to give various formulas and recurrences for these coefficients. So suppose $u = x_1 + y_1\sqrt{D}$ is the fundamental solution to $x^2 - Dy^2 = \pm 1$ and let $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$.

(a) Show that $x_{n+1} = x_1x_n + Dy_1y_n$ and $y_{n+1} = y_1x_n + x_1y_n$.

- We have $x_{n+1} + y_{n+1}\sqrt{D} = (x_1 + y_1\sqrt{D})^{n+1} = (x_1 + y_1\sqrt{D})(x_1 + y_1\sqrt{D})^n = (x_1 + y_1\sqrt{D})(x_n + y_n\sqrt{D}) = (x_1x_n + Dy_1y_n) + (y_1x_n + x_1y_n)\sqrt{D}$.
- Equating the coefficients yields the claimed relations $x_{n+1} = x_1x_n + Dy_1y_n$ and $y_{n+1} = y_1x_n + x_1y_n$.

(b) Show that both sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ satisfy the two-term recurrence relation $t_{n+2} = At_{n+1} + Bt_n$ where $A = 2x_1$ and $B = -(x_1^2 - Dy_1^2)$.

- Using (a), we have $x_{n+2} - 2x_1x_{n+1} = x_1x_{n+1} + Dy_1y_{n+1} - 2x_1x_{n+1} = Dy_1y_{n+1} - x_1x_{n+1} = Dy_1(y_1x_n + x_1y_n) - x_1(x_1x_n + Dy_1y_n) = (Dy_1^2 - x_1^2)x_n$, and thus $x_{n+2} = 2x_1x_{n+1} + (Dy_1^2 - x_1^2)x_n = Ax_{n+1} + Bx_n$ as claimed.
- Likewise, $y_{n+2} - 2x_1y_{n+1} = y_1x_{n+1} - x_1y_{n+1} = y_1(x_1x_n + Dy_1y_n) - x_1(y_1x_n + x_1y_n) = (Dy_1^2 - x_1^2)y_n$, and thus $y_{n+2} = 2x_1y_{n+1} + (Dy_1^2 - x_1^2)y_n = Ay_{n+1} + By_n$ as claimed.

(c) If $\overline{u} = x_1 - y_1\sqrt{D}$ is the conjugate of u, show that $x_n - y_n\sqrt{D} = \overline{u}^n$. Deduce that $x_n = \dfrac{u^n + \overline{u}^n}{2}$ and $y_n = \dfrac{u^n - \overline{u}^n}{2\sqrt{D}}$. [Hint: The conjugation map respects multiplication, just like complex conjugation does.]

- As noted in the hint, the conjugation map respects multiplication: explicitly, $\overline{(a + b\sqrt{D})(c + d\sqrt{D})} = \overline{(ac + Dbd) + (ad + bc)\sqrt{D}} = (ac + Dbd) - (ad + bc)\sqrt{D} = (a - b\sqrt{D})(c - d\sqrt{D})$.
- Therefore, we have $\overline{u}^n = \overline{u^n} = \overline{x_n + y_n\sqrt{D}} = x_n - y_n\sqrt{D}$ as claimed.
- Then adding and subtracting $u^n = x_n + y_n\sqrt{D}$ and $\overline{u}^n = x_n - y_n\sqrt{D}$ yields $x_n = \dfrac{u^n + \overline{u}^n}{2}$ and $y_n = \dfrac{u^n - \overline{u}^n}{2\sqrt{D}}$.

Remark: The reason that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ satisfy the kind of two-term linear recurrences given in (b) is because x_{n+2}, x_{n+1}, y_{n+2}, and y_{n+1} are all linear combinations of x_n and y_n, and so the sets $\{x_{n+2}, x_{n+1}, x_n\}$ and $\{y_{n+2}, y_{n+1}, y_n\}$ are linearly dependent as functions. Furthermore, the general theory of linear recurrences says that the solutions to a recurrence of the form $t_{n+2} = At_{n+1} + Bt_n$ are given by $t_n = c\alpha^n + d\beta^n$ where α and β are the roots of the characteristic polynomial $t^2 - At - B$ and c, d are some constants. Part (b) shows that the characteristic polynomial for both recurrences is $t^2 - 2x_1t + (x_1 - Dy_1^2)$, which factors as $(t - u)(t - \overline{u})$: this is why in (c) the results are of the form $x_n, y_n = cu^n + d\overline{u}^n$ for some constants c, d.

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4. The goal of this problem is to prove that if p is a prime congruent to 1 modulo 4, then there is always a solution to the negative Pell equation $x^2 - py^2 = -1$. As we showed, there exists a minimal solution (x_1, y_1) to $x^2 - py^2 = 1$ where x, y are positive and minimal.

   (a) Show that x_1 is odd, y_1 is even, and that $\gcd(x_1 + 1, x_1 - 1) = 2$.

   - Modulo 4, we have $x^2 - y^2 \equiv 1 \pmod 4$. Since squares are 0 or 1 mod 4, the only way we can have $x^2 - y^2 \equiv 1 \pmod 4$ is to have $x^2 \equiv 1$ and $y^2 \equiv 0$, meaning that x_1 is odd and y_1 is even.
   - Note that $\gcd(x_1 + 1, x_1 - 1)$ divides their difference, which is 2. However, since x_1 is odd, both $x_1 + 1$ and $x_1 - 1$ are even, so their gcd is in fact 2.

   (b) Show either that $x_1 - 1 = 2ps^2$, $x_1 + 1 = 2t^2$ or that $x_1 - 1 = 2s^2$ and $x_1 + 1 = 2pt^2$ for some positive integers s, t. [Hint: Use $x_1^2 - 1 = py^2$ and $\gcd(x_1 + 1, x_1 - 1) = 2$.]

   - Per the hint, we rearrange and factor $x_1^2 - py_1^2 = 1$ as $(x_1 - 1)(x_1 + 1) = py_1^2$. Since $x_1 - 1$ and $x_1 + 1$ have only a factor 2 in common, we see $x' = x_1 - 1$ and $x'' = x_1 + 1$ are relatively prime.
   - Since we have the factorization $x' \cdot x'' = p(y_1/2)^2$, and x', x'' are relatively prime, by the uniqueness of prime factorizations one of them must be a square and the other must be p times a square.
   - Thus, either $x_1 - 1 = 2ps^2$, $x_1 + 1 = 2t^2$ or $x_1 - 1 = 2s^2$ and $x_1 + 1 = 2pt^2$, as claimed.

   (c) With notation as in (b), show that if $x_1 - 1 = 2ps^2$ and $x_1 + 1 = 2t^2$ then $t^2 - ps^2 = 1$, contradicting the minimality of (x_1, y_1). Conclude in fact that there is an integer solution to $x^2 - py^2 = -1$.

   - If the first case holds then subtracting yields $2 = 2t^2 - 2ps^2$ so that $t^2 - ps^2 = 1$. But since s, t are positive integers, this cannot be true because t is smaller than x_1 by (b) but (x_1, y_1) was assumed to be minimal.
   - Thus we must have $x_1 - 1 = 2s^2$ and $x_1 + 1 = 2pt^2$ and so subtracting now gives $s^2 - pt^2 = -1$. Thus there is an integer solution to $x^2 - py^2 = -1$ as claimed.

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5. The goal of this problem is to establish some cases in which the negative Pell equation $x^2 - Dy^2 = -1$ has no solutions.

   (a) Suppose that D is divisible by 4. Show that $x^2 - Dy^2 = -1$ has no solutions.

   - Reducing modulo 4 yields $x^2 \equiv -1 \pmod 4$ which has no solutions.

   (b) Suppose that p is an odd prime and that there is a solution to the congruence $x^2 \equiv -1 \pmod p$. Prove that $p \equiv 1 \pmod 4$. [Hint: Explain why x has order 4 in the multiplicative group of nonzero residues modulo p, and then use Lagrange's theorem or Euler's theorem.]

   - Note that $x^4 \equiv (-1)^2 \equiv 1 \pmod p$, so the order of x must divide 4. However, because $x^2 \equiv -1 \pmod p$ is not 1 mod p, since p is odd, the order of x cannot divide 2, hence it must be 4.
   - Now apply Lagrange's theorem to the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ of nonzero residues modulo p: this group has order $p - 1$. But by Lagrange's theorem, the order of any element must divide the order of the group, and so 4 must divide $p - 1$, meaning that $p \equiv 1 \pmod 4$ as claimed.
   - Alternatively, by Euler's theorem, we have $x^{p-1} \equiv 1 \pmod p$, and so since $p - 1$ is even, we can write $(-1)^{(p-1)/2} = (x^2)^{(p-1)/2} = x^{p-1} \equiv 1 \pmod p$. Since $-1 \not\equiv 1 \pmod p$, we must have $(-1)^{(p-1)/2} \equiv 1 \pmod p$ and so $(p - 1)/2$ must be even.

   (c) Suppose that D is divisible by a prime that is congruent to 3 modulo 4. Show that $x^2 - Dy^2 = -1$ has no solutions.

   - Suppose $p | D$ is congruent to 3 modulo 4. Then modulo p we see that $x^2 \equiv -1 \pmod p$. But this is a contradiction by part (b), since $x^2 \equiv -1 \pmod p$ has no solutions unless $p = 2$ or $p \equiv 1 \pmod 4$.

   (d) By (a)-(c) above, if $x^2 - Dy^2 = -1$ has solutions, then D is a product of primes congruent to 1 modulo 4, possibly times 2. In fact, not all such integers do have a solution to the negative Pell equation: show specifically that $x^2 - 34y^2 = -1$ and $x^2 - 221y^2 = -1$ have no solutions.

   - We can use the super magic box to compute the fundamental unit in each case.
   - For $D = 34$ the fundamental unit is $35 + 6\sqrt{34}$: since $35^2 - 6 \cdot 34^2 = 1$, there is no solution to $x^2 - 34y^2 = -1$.
   - For $D = 221$ the fundamental unit is $1665 + 112\sqrt{221}$: since $1665^2 - 221 \cdot 112^2 = 1$, there is no solution to $x^2 - 221y^2 = -1$.

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6. The goal of this problem is to construct uncountably many transcendental numbers.

(a) Prove that the real number $\alpha = \sum_{k=1}^{\infty} \frac{1}{(k!)^{k!}} = 1 + \frac{1}{2^2} + \frac{1}{6^6} + \frac{1}{24^{24}} + \cdots$ is transcendental.

- We use Liouville's criterion. If we let $\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{1}{(k!)^{k!}}$ then it is easy to see that the denominator q_n equals $(n!)^{n!}$ since all of the other denominators in the sum divide it.
- Furthermore, the tail of the sum $\sum_{k=n+1}^{\infty} \frac{1}{(k!)^{k!}}$ is bounded above by $\frac{2}{(n+1)!^{(n+1)!}}$ since it is certainly bounded above by the geometric series $\frac{1}{2^{n+d}(n+1)!^{(n+1)!}}$.
- Therefore, we see $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{2}{(n+1)!^{(n+1)!}} < \frac{2}{(n!)^{(n+1)!}} = \frac{2}{q_n^{n+1}} < \frac{2}{q_n^n}$.
- Thus, by Liouville's theorem, this means α is transcendental, as claimed.

(b) Generalize (a) by showing that any number of the form $\beta = \sum_{k=1}^{\infty} \frac{(-1)^{s_k}}{(k!)^{k!}}$ is transcendental, for any choice of signs $(-1)^{s_k} = \pm 1$ on each term.

- We use Liouville's criterion. If we let $\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{(-1)^{s_k}}{(k!)^{k!}}$ then as in (a) the denominator q_n equals $(n!)^{n!}$ since all of the other denominators in the sum divide it.
- Furthermore, the tail of the sum $\sum_{k=n+1}^{\infty} \frac{1}{(k!)^{k!}}$ is bounded above in absolute value by $\frac{2}{(n+1)!^{(n+1)!}}$ so as before we see $\left|\beta - \frac{p_n}{q_n}\right| < \frac{2}{(n+1)!^{(n+1)!}} < \frac{2}{(n!)^{(n+1)!}} = \frac{2}{q_n^{n+1}} < \frac{2}{q_n^n}$.
- Thus, by Liouville's criterion, this means β is transcendental, as claimed.

(c) Show that all of the numbers in (b) are distinct and lie in $(-2, 2)$. Deduce that there are uncountably many transcendental numbers in the interval $(-2, 2)$. [Hint: For distinctness, suppose that two of the numbers in (b) are equal and have their first unequal signs in the dth term. Show that the tails of the series are too small to account for the difference.]

- Suppose we had an equality $\sum_{k=1}^{\infty} \frac{(-1)^{s_k}}{(k!)^{k!}} = \sum_{k=1}^{\infty} \frac{(-1)^{t_k}}{(k!)^{k!}}$ for some sign choices s_k and t_k that differ first in the dth term. Then subtracting yields $\frac{2}{(d!)^{d!}} + \sum_{k=d+1}^{\infty} \frac{(-1)^{s_k} - (-1)^{t_k}}{(k!)^{k!}} = 0$, and the tail of the series has absolute value at most $\sum_{k=d+1}^{\infty} \frac{2}{(k!)^{k!}} < \frac{2}{(d+1)!^{(d+1)!}}$, but this is strictly smaller than $\frac{2}{(d!)^{d!}}$.
- This is impossible, so all of the numbers in (b) are distinct. Observe also that all of the numbers in (b) are transcendental and that there are uncountably many of them since they are in bijection with the infinite binary decimal sequences.
- Observe that all of the numbers constructed in (b) are transcendental and also their absolute values are less or equal to the number α from (a). Since $\alpha < \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$ we see that all of them lie in the open interval $(-2, 2)$.
- Putting all of this together yields the claimed result that there are uncountably many transcendental numbers in the interval $(-2, 2)$.

(d) Conclude in fact that there are uncountably many transcendental numbers in any open interval (a, b) for any $a < b$.

- We can just rescale $(-2, 2)$ by a rational multiple and then translate by a rational amount to make it land inside (a, b). Since these two operations preserve transcendentality and $(-2, 2)$ has uncountably many transcendentals by (c), so does (a, b).

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7. [Challenge] It is a theorem of Hurwitz, mentioned in class, that if α is an arbitrary irrational number, then there exist infinitely many p/q with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2}$. The goal of this problem is to prove that $\sqrt{5}$ cannot be replaced by any larger constant (i.e., that Hurwitz's theorem is sharp). So let $C > \sqrt{5}$.

   (a) Let $\varphi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, \ldots]$ be the golden ratio and suppose that $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$. Show that $\frac{p}{q} = \frac{F_{n+1}}{F_n}$ for some positive integer n, where F_n is the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 1$.

   - Because $C > \sqrt{5} > 2$, by our results we know that any rational number with $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$ must be a continued fraction convergent to φ.
   - As we noted in class, and is easy to prove via induction, the convergents to φ are ratios of consecutive Fibonacci numbers. Thus, $\frac{p}{q} = \frac{F_{n+1}}{F_n}$ for some $n \geq 1$.

   (b) Suppose $\alpha = [a_0, a_1, a_2, \ldots]$. Show that $\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})}$.

   - Note that $\alpha = [a_0, a_1, a_2, \ldots, a_n, \alpha_{n+1}] = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}}$, and so $\alpha - \frac{p_n}{q_n} = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_n q_{n-1}}{q_n(\alpha_{n+1}q_n + q_{n-1})} = \frac{(-1)^{n-1}}{q_n(\alpha_{n+1}q_n + q_{n-1})}$.
   - Taking the absolute value then gives the desired expression immediately.

   (c) Show that $\left|\varphi - \frac{F_{n+1}}{F_n}\right| = \frac{1}{F_n^2(\varphi + F_{n-1}/F_n)}$ and that $\lim_{n\to\infty} [\varphi + F_{n-1}/F_n] = \sqrt{5}$.

   - The first statement is immediate from the identity proven in (b), since the nth convergent of φ is $\frac{F_{n+1}}{F_n}$ as shown in (a) and the remainder term α_{n+1} is always φ from the continued fraction expansion.
   - For the second part, note $\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\varphi} = \frac{\sqrt{5}-1}{2}$, so $\lim_{n\to\infty} [\varphi + \frac{F_{n-1}}{F_n}] = \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5}$.

   (d) Deduce that if $C > \sqrt{5}$, then there are only finitely many rational numbers p/q such that $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$.

   - Suppose $C > \sqrt{5}$ and that $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$. By (a), we know that $\frac{p}{q} = \frac{F_{n+1}}{F_n}$ for some n. So then by (c) we know that $\left|\varphi - \frac{F_{n+1}}{F_n}\right| = \frac{1}{F_n^2(\varphi + F_{n-1}/F_n)}$.
   - However, since the second term in the denominator has limit $\sqrt{5}$ as $n \to \infty$, there are only finitely many n for which $\varphi + F_{n-1}/F_n > C$ and therefore only finitely many n for which $\left|\varphi - \frac{F_{n+1}}{F_n}\right| < \frac{1}{CF_n^2}$. Hence there are only finitely many rational numbers with $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$, as claimed.
