E. Dummit's Math 4527 ∼ Number Theory 2, Spring 2025 ∼ Homework 2, due Tue Jan 21st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Express the following continued fractions as real numbers:

   (a) $[3, 1, 4, 1, 5]$.
   (b) $[\overline{1, 2, 3}]$.
   (c) $[\overline{3, 2, 1}]$.
   (d) $[\overline{3, 1, 2}]$.
   (e) $[3, \overline{1, 2}]$.
   (f) $[1, 2, \overline{1, 9, 1}]$.

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2. Find the continued fraction expansion, and the first five convergents, for each of the following:

   (a) $\sqrt{3}$.
   (b) $\sqrt{11}$.
   (c) $\dfrac{4+\sqrt{13}}{5}$.

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3. Find the rational number with denominator less than N closest to each of the following real numbers α:

   (a) $\alpha = \sqrt{13}$, $N = 100$.
   (b) $\alpha = \sqrt{2}$, $N = 100$.
   (c) $\alpha = e$, $N = 10000$. [Hint: See problem 8 for the continued fraction expansion of e.]

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4. Find a rational number with denominator less than the given bound that agrees with the given real number to the number of decimal places shown:

   (a) 0.2598425196850, denominator less than 1,000.
   (b) 0.876638301211979, denominator less than 100,000.
   (c) 0.0104091625364964, denominator less than 1,000,000.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Let $\alpha = [\overline{3}] = [3, 3, 3, 3, 3, 3, 3, \dots]$.

   (a) Find α.
   (b) The first convergent to α is 3. Find the next five convergents to α.
   (c) Show that the nth convergent to α is the ratio s_n/s_{n-1} where $s_0 = 1$, $s_1 = 3$, and for $n \geq 2$, $s_{n+1} = 3s_n + s_{n-1}$. Deduce $\lim_{n\to\infty} s_{n+1}/s_n = \alpha$.

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6. The goal of this problem is to give a method for manipulating continued fractions using linear algebra. So suppose $a_0, a_1, \dots, a_n, \dots$ is a sequence of positive integers and set $p_n/q_n = [a_0, a_1, \dots, a_n]$ for each n.

   (a) Prove that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (b) Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. [Hint: Determinant.]

   (c) Show that $[a_n, a_{n-1}, \dots, a_0] = p_n/p_{n-1}$ and that $[a_n, a_{n-1}, \dots, a_1] = q_n/q_{n-1}$. [Hint: Transpose.]

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7. The goal of this problem is to give another proof of Dirichlet's Diophantine approximation theorem (indeed, this is essentially Dirichlet's original proof, and it represents the first recognized use of the pigeonhole principle). Let α be an irrational real number.

   (a) Let n be a positive integer and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Show that some two of 0, $\{\alpha\}$, $\{2\alpha\}$, ... , $\{n\alpha\}$, 1 must be within $\dfrac{1}{n+1}$ of each other. [Hint: Pigeonhole.]

   (b) Let n be a positive integer. Show that there exists a positive integer $q \le n$ and an integer p such that $|q\alpha - p| < \frac{1}{n+1}$. [Hint: If $c\alpha$ and $d\alpha$ have fractional parts near each other, consider $(c-d)\alpha$.]

   (c) Show that there exist infinitely many pairs of integers (p, q) such that $|q\alpha - p| < 1/q$.

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8. Let $D > 1$ be a nonsquare integer. The goal of this problem is to show that rational approximations of $\sqrt{D}$ cannot be "too good".

   (a) Suppose that p/q is rational. Show that $\left|\sqrt{D} - \dfrac{p}{q}\right| \ge \dfrac{1}{3q^2\sqrt{D}}$. [Hint: Suppose $\left|p - q\sqrt{D}\right| < \frac{1}{3q\sqrt{D}}$. Explain why $\left|p + q\sqrt{D}\right| < 2q\sqrt{D} + \frac{1}{3q\sqrt{D}}$, then multiply these inequalities.]

   (b) Suppose that $\epsilon > 0$. Show that there are only finitely many rationals p/q such that $\left|\sqrt{D} - \dfrac{p}{q}\right| < \dfrac{1}{q^{2+\epsilon}}$.

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9. [Challenge] The goal of this problem is to obtain the continued fraction expansion $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$. Let $\beta = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \dots]$ be the real number with continued fraction terms $a_{3i} = 1$, $a_{3i+1} = 2i$, and $a_{3i+2} = 1$ for each $i \ge 0$, and let $C_i = p_i/q_i$ be its convergents.

   (a) Show that the convergents $C_i = p_i/q_i$ have numerators and denominators satisfying the recurrences

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} & q_{3n} &= q_{3n-1} + q_{3n-2} \\ p_{3n+1} &= 2np_{3n} + p_{3n-1} & q_{3n+1} &= 2nq_{3n} + q_{3n-1} \\ p_{3n+2} &= p_{3n+1} + p_{3n} & q_{3n+2} &= q_{3n+1} + q_{3n} \end{aligned}$$

   with initial values $p_0 = p_1 = q_0 = q_2 = 1$, $q_1 = 0$, and $p_2 = 2$.

   (b) Now define the integrals $A_n = \displaystyle\int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx$, $B_n = \displaystyle\int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x \, dx$, $C_n = \displaystyle\int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x \, dx$. Show that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$, and $C_n = B_n - A_n$, and also that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$. [Hint: For the first two, compute the derivatives of $\frac{1}{n!}x^n(x-1)^n e^x$ and $\frac{1}{n!}x^n(x-1)^{n+1}e^x$ and then integrate both sides from $x = 0$ to $x = 1$.]

   (c) With notation as in part (b), show that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$. [Hint: Show that A_n, B_n, C_n satisfy the same recurrences as the given combinations of p_n, q_n and also have the same initial values.]

   (d) Conclude that $\beta = \lim_{i\to\infty} p_i/q_i = e$, and from this fact deduce the continued fraction expansion of e.

Remark: This argument was originally given by Hermite, and is adapted from an article of H.A. Cohn.
