1. Express the following continued fractions as real numbers:

   (a) $[3, 1, 4, 1, 5]$.

   - Either using the convergent recurrence or just writing it out, we get $[3, 1, 4, 1, 5] = \boxed{\dfrac{134}{35}}$.

   (b) $[\overline{1, 2, 3}]$.

   - If $\alpha = [\overline{1, 2, 3}]$, then $\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\alpha}}} = \dfrac{10\alpha + 3}{7\alpha + 2}$.
   - Thus $\alpha(7\alpha + 2) = 10\alpha + 3$, so $\alpha = \dfrac{4 \pm \sqrt{37}}{7}$. Since $\alpha > 1$, we see $\alpha = \boxed{\dfrac{4 + \sqrt{37}}{7}}$.

   (c) $[\overline{3, 2, 1}]$.

   - If $\alpha = [\overline{3, 2, 1}]$, then $\alpha = 3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\alpha}}} = \dfrac{10\alpha + 7}{3\alpha + 2}$.
   - Thus $\alpha(3\alpha + 2) = 10\alpha + 7$, so $\alpha = \dfrac{4 \pm \sqrt{37}}{3}$. Since $\alpha > 3$, we see $\alpha = \boxed{\dfrac{4 + \sqrt{37}}{3}}$.

   (d) $[\overline{3, 1, 2}]$.

   - If $\alpha = [\overline{3, 1, 2}]$, then $\alpha = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\alpha}}} = \dfrac{11\alpha + 4}{3\alpha + 1}$.
   - Thus $\alpha(3\alpha + 1) = 11\alpha + 4$, so $\alpha = \dfrac{5 \pm \sqrt{37}}{3}$. Since $\alpha > 3$, we see $\alpha = \boxed{\dfrac{5 + \sqrt{37}}{3}}$.

   (e) $[3, \overline{1, 2}]$.

   - If $\alpha = [\overline{1, 2}]$, then $\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{\alpha}} = \dfrac{3\alpha + 1}{2\alpha + 1}$.
   - Thus $\alpha(2\alpha + 1) = 3\alpha + 1$, so $\alpha = \dfrac{1 \pm \sqrt{3}}{2}$. Since $\alpha > 3$, we see $\alpha = \dfrac{1 + \sqrt{3}}{2}$.
   - Then $[3, \overline{1, 2}] = 3 + \dfrac{1}{\alpha} = \boxed{2 + \sqrt{3}}$.

   (f) $[1, 2, \overline{1, 9, 1}]$.

   - If $\alpha = [\overline{1, 9, 1}]$, then $\alpha = 1 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{\alpha}}} = \dfrac{11\alpha + 10}{10\alpha + 9}$.
   - Thus $\alpha(11 + 10\alpha) = 10\alpha + 9$, so $\alpha = \dfrac{1 \pm \sqrt{101}}{10}$. Since $\alpha > 1$, we see $\alpha = \dfrac{1 + \sqrt{101}}{10}$.
   - Then $[1, 2, \overline{1, 9, 1}] = 1 + \cfrac{1}{2 + \cfrac{1}{\alpha}} = \boxed{\dfrac{45 - \sqrt{101}}{26}}$.

---

2. Find the continued fraction expansion, and the first five convergents, for each of the following:

(a) $\sqrt{3}$.

- We use the algorithm described in the notes: with $\alpha = \sqrt{3}$, we set $a_0 = \lfloor \alpha \rfloor$, and then for each $i \geq 1$ we take $\alpha_i = \dfrac{1}{\alpha_{i-1} - a_{i-1}}$, and $a_i = \lfloor \alpha_i \rfloor$.
- With $\alpha = \sqrt{3}$, we find, successively,

n	0	1	2	$\cdots$
α_n	$\sqrt{3}$	$\dfrac{\sqrt{3}+1}{2}$	$\sqrt{3}+1$	$\cdots$
a_n	1	1	2	$\cdots$
$\alpha_n - a_n$	$\sqrt{3}-1$	$\dfrac{\sqrt{3}-1}{2}$	$\sqrt{3}-1$	$\cdots$

and since each term after this will repeat, we see that $\sqrt{3} = \boxed{[1, \overline{1, 2}]}$. The first 5 convergents are $[1] = 1$, $[1,1] = 2$, $[1,1,2] = 5/3$, $[1,1,2,1] = 7/4$, $[1,1,2,1,2] = 19/11$.

(b) $\sqrt{11}$.

- With $\alpha = \sqrt{11}$, we find, successively,

n	0	1	2	$\cdots$
α_n	$\sqrt{11}$	$\dfrac{\sqrt{11}+3}{2}$	$\sqrt{11}+3$	$\cdots$
a_n	3	3	6	$\cdots$
$\alpha_n - a_n$	$\sqrt{11}-3$	$\dfrac{\sqrt{11}-3}{2}$	$\sqrt{11}-3$	$\cdots$

and since each term after this will repeat, we see that $\sqrt{11} = \boxed{[3, \overline{3, 6}]}$. The first 5 convergents are $[3] = 3$, $[3,3] = 10/3$, $[3,3,6] = 63/19$, $[3,3,6,3] = 199/60$, $[3,3,6,3,6] = 1257/379$.

(c) $\dfrac{4+\sqrt{13}}{5}$.

- With $\alpha = \dfrac{4+\sqrt{13}}{5}$, we find, successively,

n	0	1	2	3	4	$\cdots$
α_n	$\dfrac{\sqrt{13}+4}{5}$	$\dfrac{5\sqrt{13}+5}{12}$	$\dfrac{5\sqrt{13}+7}{23}$	$\dfrac{5\sqrt{13}+16}{3}$	$\dfrac{5\sqrt{13}+17}{12}$	$\cdots$
a_n	1	1	1	11	2	$\cdots$
$\alpha_n - a_n$	$\dfrac{\sqrt{13}-1}{5}$	$\dfrac{5\sqrt{13}-7}{12}$	$\dfrac{5\sqrt{13}-16}{23}$	$\dfrac{5\sqrt{13}-17}{3}$	$\dfrac{5\sqrt{13}-7}{12}$	$\cdots$

and since each term after this will repeat, we see that $\dfrac{4+\sqrt{13}}{5} = \boxed{[1, 1, \overline{1, 11, 2}]}$. The first 5 convergents are $[1] = 1$, $[1,1] = 2$, $[1,1,1] = 3/2$, $[1,1,1,11] = 35/23$, $[1,1,1,11,2] = 73/48$.

---

3. Find the rational number with denominator less than N closest to each of the following real numbers α:

   (a) $\alpha = \sqrt{13}$, $N = 100$.

   - We have the continued fraction expansion $\sqrt{13} = [3, \overline{1, 1, 1, 1, 6}]$, so the first few convergents are 3, 4, 7/2, 11/3, 18/5, 119/33, 137/38, 256/71.
   - The last two convergents with denominator less than 100 are 137/38 and 256/71. There are no terms of the Farey sequence of level 99 between them.
   - We compute $\sqrt{13} - 137/38 \approx 2.8812 \cdot 10^{-4}$ and $\sqrt{13} - 256/71 \approx -8.2528 \cdot 10^{-5}$ so the best approximation is $\boxed{256/71}$.

   (b) $\alpha = \sqrt{2}$, $N = 100$.

   - We have the continued fraction expansion $\sqrt{2} = [1, \overline{2}]$, so the first few convergents are 1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, and so on.
   - The last two convergents with denominator less than 100 are 41/29 and 99/70. The only term between them in the Farey sequence of level 99 is their mediant, 140/99.
   - We can then compute that $\sqrt{2} - 41/29 \approx 4.205 \cdot 10^{-4}$, $\sqrt{2} - 140/99 \approx 7.2148 \cdot 10^{-5}$, and $\sqrt{2} - 99/70 \approx -7.2152 \cdot 10^{-5}$. Thus, the best approximation is $\boxed{140/99}$ (but just barely!).

   (c) $\alpha = e$, $N = 10000$. [Hint: See problem 9 for the continued fraction expansion of e.]

   - From problem 7 we have the continued fraction expansion $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$ so the first few convergents are 2, 3, 8/3, 11/4, 19/7, 87/32, 193/71, 1264/465, 1457/536, 2721/1001, 23225/8544, 25946/9545, 49171/18089, and so on.
   - The last two convergents with denominator less than 10000 are 23225/8544 and 25946/9545. Since their mediant is actually the next term 49171/18089, there are no terms between them in the Farey sequence of level 10000.
   - We can then compute $e - 23225/8544 \approx -6.7469 \cdot 10^{-9}$ while $e - 25946/9545 \approx 5.5151 \cdot 10^{-9}$, so the best approximation is $\boxed{25946/9545}$.

---

4. Find a rational number with denominator less than the given bound that agrees with the given real number to the number of decimal places shown:

   (a) 0.2598425196850, denominator less than 1,000.

   - We have $0.2598425196850 = [0, 3, 1, 5, 1, 1, 1, 1, 1574803149]$.
   - We truncate the enormous final term to obtain $[0, 3, 1, 5, 1, 1, 1, 1] = \boxed{33/127}$.
   - Computing the decimal expansion shows it is indeed equal to 0.2598425196850....

   (b) 0.876638301211979, denominator less than 100,000.

   - We have $0.876638301211979 = [0, 1, 7, 9, 2, 2, 2, 1, 5, 1, 1, 4, 1, 261056, 1, 4, 2, 1, 2, 1, 5, \ 4, 4, 1, 5]$.
   - We truncate ahead of the very large middle term to obtain $[0, 1, 7, 9, 2, 2, 2, 1, 5, 1, 1, 4, 1] = \boxed{78190/89193}$.
   - Computing the decimal expansion shows it is indeed equal to 0.876638301211979....

   (c) 0.0104091625364964, denominator less than 1,000,000.

   - We have $0.0104091625364964 = [0, 96, 14, 2, 4, 2, 2, 1, 2, 1, 1, 1, 12950, 1, 1, 3, 11, 1, 23, 2, 3, 6, 2]$.
   - We truncate ahead of the very large middle term to obtain $[0, 96, 14, 2, 4, 2, 2, 1, 2, 1, 1, 1] = \boxed{10100/970299}$.
   - Computing the decimal expansion shows it is indeed equal to 0.0104091625364964....

---

5. Let $\alpha = [\overline{3}] = [3,3,3,3,3,3,3,\ldots]$.

   (a) Find α.

   - We have $\alpha = 3 + 1/\alpha$ so $\alpha^2 - 3\alpha - 1 = 0$ so since $3 < \alpha < 4$ we have $\alpha = \boxed{(3+\sqrt{13})/2}$.

   (b) The first convergent to α is 3. Find the next five convergents to α.

   - Using the magic box we obtain 10/3, 33/10, 109/33, 360/109, 1189/360.

   (c) Show that the nth convergent to α is the ratio s_n/s_{n-1} where $s_0 = 1$, $s_1 = 3$, and for $n \geq 2$, $s_{n+1} = 3s_n + s_{n-1}$. Deduce $\lim_{n\to\infty} s_{n+1}/s_n = \alpha$.

   - The numerator and denominator recurrences are $p_{n+1} = 3p_n + p_{n-1}$ and $q_{n+1} = 3q_n + q_{n-1}$. Since $p_0 = 1$, $p_1 = 3$, $p_2 = 10$ while $q_0 = 0$, $q_1 = 1$, $q_2 = 3$ we see that $p_0 = q_1 = s_0$ and $p_1 = q_2 = a_1$ so by induction since the sequences satisfy the same recurrences and have the same initial conditions, we see $p_n = s_n$ and $q_n = s_{n-1}$ for all n.
   - The last part follows immediately from the fact that the sequence of convergents converges to α.

---

6. The goal of this problem is to give a method for manipulating continued fractions using linear algebra. So suppose $a_0, a_1, \ldots, a_n, \ldots$ is a sequence of positive integers and set $p_n/q_n = [a_0, a_1, \ldots, a_n]$ for each n.

   (a) Prove that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$.

   - First, we have $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n p_{n-1} + p_{n-2} & a_n q_{n-1} + q_{n-2} \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{bmatrix}$, which is simply a rewriting of the recurrence relation $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$.
   - Therefore, by a trivial induction, we see that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$, since $\begin{bmatrix} p_1 & q_1 \\ p_0 & q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

   (b) Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. [Hint: Determinant.]

   - Take the determinant of the identity from (a): each of the matrices on the right has determinant -1, so by the multiplicativity of the determinant we immediately deduce that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$.

   (c) Show that $[a_n, a_{n-1}, \ldots, a_0] = p_n/p_{n-1}$ and that $[a_n, a_{n-1}, \ldots, a_1] = q_n/q_{n-1}$. [Hint: Transpose.]

   - Take the transpose of the identity from (a): each of the matrices on the right is symmetric, so the transpose is simply the product in reverse order.
   - We obtain $\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$.
   - But by part (a), the right-hand side is equal to $\begin{bmatrix} p'_n & p'_{n-1} \\ q'_n & q'_{n-1} \end{bmatrix}$, where p'_0/q'_0, p'_1/q'_1, $\ldots$, p'_n/q'_n are the convergents to the continued fraction $[a_n, a_{n-1}, \ldots, a_0]$ with terms in reverse order. Thus, by part (a) again, its nth convergent $[a_n, a_{n-1}, \ldots, a_0]$ must equal p_n/p_{n-1} and its $(n-1)$st convergent $[a_n, a_{n-1}, \ldots, a_1]$ must equal q_n/q_{n-1}.

---

7. The goal of this problem is to give another proof of Dirichlet's Diophantine approximation theorem (indeed, this is essentially Dirichlet's original proof, and it represents the first recognized use of the pigeonhole principle). Let α be an irrational real number.

   (a) Let n be a positive integer. Show that the fractional parts of some two of of 0, α, 2α, ... , $n\alpha$ must be within $\dfrac{1}{n+1}$ of each other. [Hint: Pigeonhole, paying attention to the intervals near 0.]

   - Following the hint, consider the $n+1$ intervals $[0, \frac{1}{n+1})$, $[\frac{1}{n+1}, \frac{2}{n+1})$, ... , $[\frac{n}{n+1}, 1)$.
   - Each of the n fractional parts lands in one of these intervals. If there is a term in either of the intervals at the end, then it is a distance less than $\frac{1}{n+1}$ from zero.
   - Otherwise, by the pigeonhole principle, we have n nonzero fractional parts that land in $n-1$ intervals, so some two must be in the same interval hence have a difference less than $\frac{1}{n+1}$.

   (b) Let n be a positive integer. Show that there exists a positive integer $q \le n$ and an integer p such that $|q\alpha - p| < \frac{1}{n+1}$. [Hint: If $c\alpha$ and $d\alpha$ have fractional parts near each other, consider $(c-d)\alpha$.]

   - By (a), the fractional parts of some $c\alpha$ and $d\alpha$ for $0 \le c, d \le n$ must differ by less than $\frac{1}{n+1}$, which is to say, the number $c\alpha - d\alpha = (c-d)\alpha$ differs from an integer by less than $\frac{1}{n+1}$.
   - Taking $q = |c-d|$ and taking p to be that integer (with appropriate sign) we see that $|q\alpha - p| < \frac{1}{n+1}$ as required.

   (c) Show that there exist infinitely many pairs of integers (p, q) such that $|q\alpha - p| < 1/q$.

   - Apply (b) to an increasing sequence of values of n; since $q \le n$ we see $|q\alpha - p| < \frac{1}{n+1} < \frac{1}{q}$ as needed.
   - Now, since α is irrational, each of the values $|q\alpha - p|$ is positive, and so for any finite list of pairs (p, q), we can use (b) to construct a new one by choosing n large enough to force $|q\alpha - p|$ to be less than the corresponding value for all of the pairs (p, q) on our list.
   - Thus, there exist infinitely many pairs of integers (p, q) such that $|q\alpha - p| < 1/q$, as claimed.

---

8. Let $D > 1$ be a nonsquare integer. The goal of this problem is to show that rational approximations of $\sqrt{D}$ cannot be "too good".

   (a) Suppose that p/q is rational. Show that $\left|\sqrt{D} - \dfrac{p}{q}\right| \ge \dfrac{1}{3q^2\sqrt{D}}$. [Hint: Suppose $\left|p - q\sqrt{D}\right| < \frac{1}{3q\sqrt{D}}$. Explain why $\left|p + q\sqrt{D}\right| < 2q\sqrt{D} + \frac{1}{3q\sqrt{D}}$, then multiply these inequalities.]

   - Suppose by way of contradiction that $\left|\sqrt{D} - \frac{p}{q}\right| < \frac{1}{3q^2\sqrt{D}}$ so that upon multiplying by q we have $\left|p - q\sqrt{D}\right| < \frac{1}{3q\sqrt{D}}$. This is equivalent to $-\frac{1}{3q\sqrt{D}} < p - q\sqrt{D} < \frac{1}{3q\sqrt{D}}$ hence adding $2q\sqrt{D}$ and then taking the absolute value yields $\left|p + q\sqrt{D}\right| < 2q\sqrt{D} + \frac{1}{3q\sqrt{D}}$.
   - Now multiplying $\left|p - q\sqrt{D}\right| < \dfrac{1}{3q\sqrt{D}}$ by $\left|p + q\sqrt{D}\right| < 2q\sqrt{D} + \dfrac{1}{3q\sqrt{D}}$ yields $\left|p^2 - Dq^2\right| < \dfrac{2}{3} + \dfrac{1}{9q^2D}$ which is certainly less than 1.
   - But this is impossible because $p^2 - Dq^2$ is an integer and it cannot be zero because $\sqrt{D}$ is irrational.

   (b) Suppose that $\epsilon > 0$. Show that there are only finitely many rationals p/q such that $\left|\sqrt{D} - \dfrac{p}{q}\right| < \dfrac{1}{q^{2+\epsilon}}$.

   - By (a) we have $\left|\sqrt{D} - \dfrac{p}{q}\right| \ge \dfrac{1}{3q^2\sqrt{D}}$. If this must be less than $\dfrac{1}{q^{2+\epsilon}}$ then we must have $q^{2+\epsilon} < 3q^2\sqrt{D}$ which forces $q < (3\sqrt{D})^{1/\epsilon}$ which is finite.
   - Thus there are only finitely many possible q and so there are only finitely many possible p/q satisfying the inequality.

---

9. [Challenge] The goal of this problem is to obtain the continued fraction expansion $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$. Let $\beta = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \dots]$ be the real number with continued fraction terms $a_{3i} = 1$, $a_{3i+1} = 2i$, and $a_{3i+2} = 1$ for each $i \geq 0$, and let $C_i = p_i/q_i$ be its convergents.

   (a) Show that the convergents $C_i = p_i/q_i$ have numerators and denominators satisfying the recurrences

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} & q_{3n} &= q_{3n-1} + q_{3n-2} \\ p_{3n+1} &= 2np_{3n} + p_{3n-1} & q_{3n+1} &= 2nq_{3n} + q_{3n-1} \\ p_{3n+2} &= p_{3n+1} + p_{3n} & q_{3n+2} &= q_{3n+1} + q_{3n} \end{aligned}$$

   with initial values $p_0 = p_1 = q_0 = q_2 = 1$, $q_1 = 0$, and $p_2 = 2$.

   - These all follow from the recursive definitions $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$. The initial values are also immediate from the initial expansion: $p_0/q_0 = [1] = 1/1$, $p_1/q_1 = [1, 0] = 1/0$, $p_2/q_2 = [1, 0, 1] = 2/1$.

   (b) Now define the integrals $A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx$, $B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x \, dx$, $C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x \, dx$. Show that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$, and $C_n = B_n - A_n$, and also that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$. [Hint: For the first two, compute the derivatives of $\frac{1}{n!}x^n(x-1)^n e^x$ and $\frac{1}{n!}x^n(x-1)^{n+1}e^x$ and then integrate both sides from $x = 0$ to $x = 1$.]

   - Following the hint, we compute $\frac{d}{dx}\left[\frac{1}{n!}x^n(x-1)^n e^x\right] = \frac{1}{(n-1)!}x^{n-1}(x-1)^n e^x + \frac{1}{(n-1)!}x^n(x-1)^{n-1}e^x + \frac{1}{n!}x^n(x-1)^n e^x$. Integrating both sides from 0 to 1 yields $C_{n-1} + B_{n-1} + A_n = \frac{1}{n!}x^n(x-1)^n e^x\big|_{x=0}^1 = 0$, and so $A_n = -B_{n-1} - C_{n-1}$.
   - Similarly, we also have $\frac{d}{dx}\left[\frac{1}{n!}x^n(x-1)^{n+1}e^x\right] = \frac{1}{(n-1)!}x^{n-1}(x-1)^{n+1}e^x + \frac{n+1}{n!}x^n(x-1)^n e^x + \frac{1}{n!}x^n(x-1)^{n+1}e^x$. The first term is $\frac{1}{(n-1)!}x^{n-1}(x-1)^{n-1}[x(x-1)-(x-1)]e^x = \frac{n}{n!}x^n(x-1)^n e^x - \frac{1}{(n-1)!}x^{n-1}(x-1)^n e^x$. Thus, integrating both sides from 0 to 1 yields $nA_n - B_n + (n+1)A_n + C_n = 0$, so that $B_n = -2nA_n + C_{n-1}$.
   - Finally, for the third relation, we have $x^n(x-1)^n + x^n(x-1)^{n+1} = x^{n+1}(x-1)^n$, so scaling appropriately and integrating yields $A_n + C_n = B_n$ and thus $C_n = B_n - A_n$.
   - For the limits as $n \to \infty$, on $[0, 1]$ note that $\left|\frac{x^n(x-1)^2}{n!}e^x\right| \leq \frac{e}{n!}$ since all the other terms are between 0 and 1 on $[0, 1]$. Therefore, $|A_n| \leq \frac{e}{n!} \to 0$ as $n \to \infty$. The integrands in B_n and C_n are also bounded above by $\frac{e}{n!}$ in the same way, so their limits are also 0 as $n \to \infty$.

   (c) With notation as in part (b), show that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$. [Hint: Show that A_n, B_n, C_n satisfy the same recurrences as the given combinations of p_n, q_n and also have the same initial values.]

   - Note $A_0 = \int_0^1 e^x \, dx = e^1 - e^0 = e - 1$, $B_0 = \int_0^1 xe^x \, dx = (xe^x - e^x)\big|_{x=0}^1 = 1$, and $C_0 = \int_0^1 (x-1)e^x \, dx = (xe^x - 2e^x)\big|_{x=0}^1 = 2 - e$.
   - By (b) we see that A_n, B_n, C_n satisfy the same recurrences as p_n, q_n and hence also any linear combination of p_n, q_n. By the calculation above we see that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$ all hold when $n = 0$, so by a trivial induction, we obtain the relations above.

   (d) Conclude that $\beta = \lim_{i\to\infty} p_i/q_i = e$, and from this fact deduce the continued fraction expansion of e.

   - Note $\frac{p_{3n}}{q_{3n}} - e = \frac{A_n}{q_{3n}} \to 0$, $\frac{p_{3n+1}}{q_{3n+1}} - e = -\frac{B_n}{q_{3n+1}} \to 0$, and $\frac{p_{3n+2}}{q_{3n+2}} - e = -\frac{C_n}{q_{3n+2}} \to 0$. Thus $\lim_{i\to\infty} \frac{p_i}{q_i} = e$.
   - The continued fraction of e is therefore given by the same expansion as β, except with the zero term cleaned up: this yields $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$ as claimed.

Remark: This argument was originally given by Hermite, and is adapted from an article of H.A. Cohn.
