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1. Find all solutions in integers (if any) to the following linear Diophantine equations:

   (a) $22a + 17b = 19$.

   - Since $\gcd(22, 17) = 1$ divides 19, there is a solution.
   - Reducing both sides modulo 17 yields $5a \equiv 2 \pmod{17}$. Since $5^{-1} \equiv 7 \pmod{17}$, scaling by 7 yields $a \equiv 14 \pmod{17}$.
   - The general solution is $(a, b) = \boxed{(14 + 17k, -17 - 22k)}$ for $k \in \mathbb{Z}$.

   (b) $35a + 77b = 24$.

   - Since $\gcd(35, 77) = 7$ does not divide 24, there is $\boxed{\text{no solution}}$.

   (c) $42a + 27b = 39$.

   - Since $\gcd(42, 27) = 3$ divides 39, there is a solution. Divide by 3 to get $14a + 9b = 13$.
   - Reducing both sides modulo 9 yields $5a \equiv -5 \pmod 9$, which clearly has the solution $a \equiv -1 \pmod 9$.
   - The general solution is $(a, b) = \boxed{(-1 + 9k, 3 + 14k)}$ for $k \in \mathbb{Z}$.

   (d) $3a + 7b + 16c = 8$.

   - Rewrite as $3(a + 2b + 5c) + b + 2c = 8$.
   - Now substitute $w = a + 2b + 5c$ to obtain the equation $3w + b + 2c = 8$, which we can easily solve as $b = -3w - 2c + 8$.
   - Substituting back yields $a = w - 2b - 5c = -16 - c + 7w$.
   - So the general solution is $(a, b, c) = \boxed{(-16 - c + 7w,\ -3w - 2c + 8,\ c)}$ for $c, w \in \mathbb{Z}$.

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2. Find all right triangles having one leg of length 40, and whose other two side lengths are integers.

   - Any such right triangle has legs of lengths $k(2st)$ and $k(s^2 - t^2)$, with hypotenuse $k(s^2 + t^2)$, where $s > t$ are unique positive integers of opposite parity and k is some unique positive integer.
   - If $40 = 2stk$, then $20 = stk$, so $(s, t, k) = (20, 1, 1)$ or $(10, 1, 2)$ or $(5, 4, 1)$ or $(5, 2, 2)$ or $(4, 1, 5)$ or $(2, 1, 10)$, yielding 40-399-401, 40-198-202, 9-40-41, 40-42-58, 40-75-85, and 30-40-50.
   - If $40 = k(s^2 - t^2)$, then k must be divisible by 8. Since $k \neq 40$ we see $k = 8$, and then $s^2 - t^2 = 5$ requires $s = 3$ and $t = 2$. This yields a 40-96-104 triangle.
   - Hence there are seven: $\boxed{\text{9-40-41}}$, $\boxed{\text{30-40-50}}$, $\boxed{\text{40-42-58}}$, $\boxed{\text{40-75-85}}$, $\boxed{\text{40-96-104}}$, $\boxed{\text{40-198-202}}$, and $\boxed{\text{40-399-401}}$.

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3. Byzantine Basketball is like regular basketball except that foul shots are worth a points instead of two points and field shots are worth b points instead of three points. Moreover, in Byzantine Basketball there are exactly 35 scores that never occur in a game, one of which is 58. What are a and b?

   - Clearly a and b must be relatively prime, or else there are infinitely many unattainable scores. Assume $a < b$.
   - By Sylvester's theorem, the number of unattainable scores is $\frac{1}{2}(a-1)(b-1)$, so $(a-1)(b-1) = 2 \cdot 35 = 2 \cdot 5 \cdot 7$.
   - Then $(a, b) = (2, 71)$, $(3, 36)$, $(6, 15)$, or $(8, 11)$. However, the middle two pairs are not relatively prime, and the first pair leaves 58 attainable.
   - Hence $(a, b) = \boxed{(8, 11) \text{ or } (11, 8)}$.

   Remark: This problem was on the 1971 Putnam exam.

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4. Compute the following things:

(a) Show that $7/13$ and $13/24$ are adjacent in the Farey sequence of level 24. What are the next three terms after them?

- We have $13 \cdot 13 - 7 \cdot 24 = 169 - 168 = 1$ so these terms are indeed consecutive in the Farey sequence of level $\max(13, 24) = 24$.
- Using the recurrence, the next three terms are $\boxed{6/11,\ 11/20,\ 5/9}$.

(b) Find all n such that exactly 2 terms appear between $7/13$ and $13/24$ in the Farey sequence of level n.

- The first term that will appear between them is the mediant $20/37$ in the Farey sequence of level 37.
- Between $7/13$ and $20/37$ the next term will be $27/50$, while between $20/37$ and $13/24$, the next intermediate term will be $33/61$.
- Thus, the first time there are two intermediate terms occurs with $n = 50$: the terms are $7/13$, $27/50$, $20/37$, $13/24$. The next term that will appear between any of these is $33/61$, and so for $\boxed{n = 50, 51, \ldots, 60}$ there are exactly two intermediate terms.

(c) List all the terms between $6/19$ and $5/14$ in the Farey sequence of level 19.

- These terms are not consecutive because $5 \cdot 19 - 6 \cdot 14 = 11$ is not 1. To generate a term between them we try the mediant, which is $(6 + 5)/(19 + 14) = 11/33 = 1/3$.
- We can then see $6/19$ and $1/3$ are consecutive since $19 - 6 \cdot 3 = 1$. Using the two-term recurrence to generate the next terms in this sequence produces $6/17$ and then $5/14$. Thus, the full list is $\boxed{6/19,\ 1/3,\ 6/17,\ 5/14}$. (We can check this by noting that $bc - ad = 1$ for each consecutive pair, and there are no missing terms because the mediants all have larger denominators.)
- Alternatively, we could just have computed the next term after $6/19$ directly, and then used the recurrence.

(d) Find the three terms following $154/227$ in the Farey sequence of level 2025.

- By the above results, if $154/227$ and c/d are consecutive terms, then $227c - 154d = 1$.
- Solving this Diophantine equation using the Euclidean algorithm produces the solutions $(c, d) = (173 + 154k,\ 255 + 227k)$ for $k \in \mathbb{Z}$.
- The larger the value of k is, the smaller the value of $\frac{c}{d} - \frac{154}{227} = \frac{1}{227d}$ will be. The largest possible value for k is $k = 7$, so the first term is $\frac{1251}{1844}$.
- Now we can apply the two-term recursion to find the next terms, which are $1097/1617$ and $943/1390$.
- Thus, the three terms are $\boxed{\frac{1251}{1844}, \frac{1097}{1617}, \frac{943}{1390}}$.

(e) List all the terms between $1502/1801$ and $1492/1789$ in the Farey sequence of level 2025.

- These terms are not consecutive since $1801 \cdot 1492 - 1502 \cdot 1789 = 14$ is not equal to 1.
- Taking the mediant gives $(1502 + 1492)/(1801 + 1789) = 1497/1795$. This term is adjacent to $1502/1801$ but not to $1492/1789$. Taking the mediant again yields $427/512$, which is adjacent to both.
- To generate terms between $1502/1801$ and $1497/1795$, taking the mediant does not work since it does not reduce to give a smaller denominator. Instead what we can do is compute the next term in the sequence following $1502/1801$ by solving $1801c - 1502b = 1$, which has solution $(c, d) = (643 + 1502k, 771 + 1801k)$ for $k \in \mathbb{Z}$.
- Taking $k = 0$ yields the term $643/771$. Then using the 2-term recurrence we can compute the next terms, which are $1070/1283$ and $1497/1795$.
- Thus, the terms are $\boxed{\frac{1502}{1801}, \frac{643}{771}, \frac{1070}{1283}, \frac{1497}{1795}, \frac{427}{512}, \frac{1492}{1789}}$ since there are no terms in the Farey sequence of level 2025 that are between these.

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5. Find the continued fraction expansions for the following rational numbers:

(a) 355/113.

- Using the Euclidean algorithm we write

$$\begin{aligned} 355 &= 3 \cdot 113 + 16 \\ 113 &= 7 \cdot 16 + 1 \\ 16 &= 16 \cdot 1 \end{aligned}$$

so $355/113 = \boxed{[3, 7, 16]}$.

(b) 418/2021.

- Using the Euclidean algorithm we write

$$\begin{aligned} 418 &= 0 \cdot 2021 + 418 \\ 2021 &= 4 \cdot 418 + 349 \\ 418 &= 1 \cdot 349 + 69 \\ 349 &= 5 \cdot 69 + 4 \\ 69 &= 17 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 \end{aligned}$$

so $418/2021 = \boxed{[0, 4, 1, 5, 17, 4]}$.

(c) 13579/2468.

- Using the Euclidean algorithm we write

$$\begin{aligned} 13579 &= 5 \cdot 2468 + 1239 \\ 2468 &= 1 \cdot 1239 + 1229 \\ 1239 &= 1 \cdot 1229 + 10 \\ 1229 &= 122 \cdot 10 + 9 \\ 10 &= 1 \cdot 9 + 1 \\ 9 &= 9 \cdot 1 \end{aligned}$$

so $13579/2468 = \boxed{[5, 1, 1, 122, 1, 9]}$.

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6. Prove that the area of any right triangle with integer side lengths is always divisible by 6, but not necessarily any integer greater than 6.

- From our characterization we can see that the area of a Pythagorean right triangle is of the form $\frac{1}{2} \cdot 2kst \cdot k(s^2 - t^2) = k^2 st(s^2 - t^2)$ where s and t are relatively prime integers of opposite parity.
- Since one of s, t is even, the area is also even.
- If neither s nor t is divisible by 3, then they are either 1 or 2 mod 3, so their squares are both 1 mod 3. But then $s^2 - t^2$ is divisible by 3. Thus, the area is also divisible by 3.
- Since the area is divisible by both 2 and 3, it is a multiple of 6.
- The 3-4-5 triangle has area 6, so clearly the area need not be divisible by anything greater than 6.

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7. The goal of this problem is to discuss the Weierstrass substitution $t = \tan(\theta/2)$. Recall the trigonometric identities $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ shown in class. Let $t = \tan(\theta/2)$.

   (a) Show that $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, and $d\theta = \frac{2dt}{1+t^2}$.

   - By (a) we have $t = \frac{\sin\theta}{1+\cos\theta}$ and $t = \frac{1-\cos\theta}{\sin\theta}$, yielding the linear system $\sin\theta - t\cos\theta = t$ and $t\sin\theta + \cos\theta = 1$. Multiplying the second equation by t and adding yields $(1+t^2)\sin\theta = 2t$ so $\sin\theta = \frac{2t}{1+t^2}$ and back-substituting yields $\cos\theta = \frac{1-t^2}{1+t^2}$.
   - Finally, since $\theta = 2\tan^{-1} t$, differentiating yields $d\theta = \frac{2dt}{1+t^2}$.

   (b) Use the Weierstrass substitution to compute the indefinite integral $\int \frac{1}{5+3\cos\theta}\,d\theta$.

   - By (a), with $t = \tan(\theta/2)$ we have $\cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2}{1+t^2}dt$. Substituting yields $\int \frac{1}{5+3\cos\theta}\,d\theta = \int \frac{1}{5+3\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{5(1+t^2)+3(1-t^2)}\,dt = \int \frac{1}{4+t^2}\,dt = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$.
   - Thus, the original integral is $\boxed{\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan(\theta/2)\right) + C}$.

   (c) Use the Weierstrass substitution to compute the definite integral $\int_{-\pi/2}^{\pi/2} \frac{1}{4-\sin\theta}\,d\theta$.

   - Substituting $t = \tan(\theta/2)$ yields $\int_{-\pi/2}^{\pi/2} \frac{1}{4-\sin\theta}\,d\theta = \int_{-1}^{1} \frac{1}{4-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int_{-1}^{1} \frac{1}{2t^2-t+2}\,dt = \int_{-1}^{1} \frac{1}{(4t-1)^2/15+1}\,dt = \frac{2}{\sqrt{15}}\tan^{-1}\frac{4t-1}{\sqrt{15}}\Big|_{t=-1}^{1} = \boxed{\frac{\pi}{\sqrt{15}}}$.

   Remark: As can be seen from (b) and (c), the Weierstrass substitution allows evaluation of any integral that is a rational function of $\sin\theta$ and $\cos\theta$.

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8. Find the smallest positive integer n such that for all integers m with $0 < m < 2024$, there exists an integer k with $\frac{m}{2024} < \frac{k}{n} < \frac{m+1}{2025}$. (Make sure to prove that your value is the smallest possible.)

   - The answer is $n = 4049$. From our results on Farey fractions, we know the mediant $\frac{2m+1}{4049}$ is always between $\frac{m}{2024}$ and $\frac{m+1}{2025}$. Thus, $n = 4047$ will work.
   - On the other hand, if we take $m = 2023$, then we require $\frac{2023}{2024} < \frac{k}{n} < \frac{2024}{2025}$. Since $2024 \cdot 2024 - 2023 \cdot 2025 = 1$, the terms $\frac{2023}{2024}$ and $\frac{2024}{2025}$ are consecutive in the Farey sequence, so the next term that appears between then is $\frac{4047}{4049}$. This means $n \geq 4049$, and so indeed $n = 4049$ is the smallest possible n.

   Remark: This is a variation of problem B1 from the 1993 Putnam exam.

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9. For each Farey fraction a/b, define $\mathcal{C}(a/b)$ to be the circle in the upper-half of the Cartesian plane of radius $r_{a/b} = 1/(2b^2)$ that is tangent to the x-axis at the point $(a/b, 0)$. These circles are called the Ford circles.

   (a) If a/b and c/d are adjacent entries in some Farey sequence, prove that the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent.

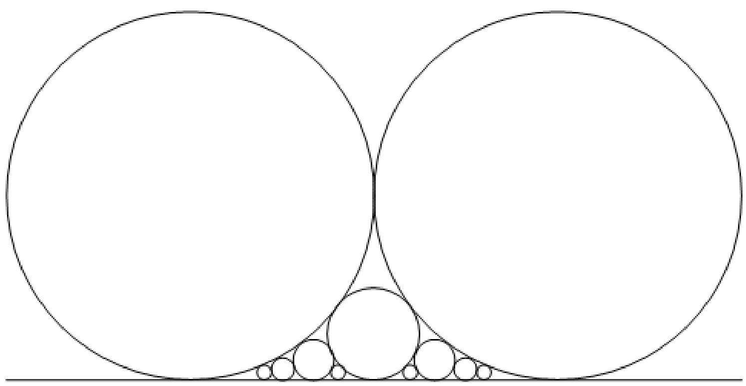
   - By our results on the Farey sequences, we know that a/b and c/d are adjacent if and only if $bc-ad = 1$.
   - The centers of the two circles are $(a/b, r_{a/b})$ and $(c/d, r_{c/d})$, so the tangency condition is equivalent to saying that $(r_{c/d}-r_{a/b})^2+(a/b-c/d)^2 = (r_{c/d}+r_{a/b})^2$, which is equivalent to $(a/b-c/d)^2 = 4r_{a/b}r_{c/d}$.
   - Since $r_{a/b} = 1/(2b^2)$ and $r_{c/d} = 1/(2d^2)$, both sides are equal to $\frac{1}{b^2d^2}$, and so the circles are indeed tangent.

   (b) If a/b and c/d are not adjacent in any Farey sequence, prove that the interiors of the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are disjoint.

   - Using the calculation in (a), we can see that the distance between the centers is equal to $(r_{c/d} - r_{a/b})^2 + (a/b - c/d)^2$, which will exceed $r_{c/d} + r_{a/b}$ whenever $|bc - ad| > 1$. Thus, if the terms are not adjacent, the interiors are disjoint.

   (c) Draw (you should probably use a computer) the 11 Ford circles corresponding to the Farey sequence of level 5.

   - Here's a plot:



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10. [Challenge] Let a, b, c be pairwise relatively prime positive integers. Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zab$ for nonnegative integers x, y, z. [Hint: Any integer is congruent modulo a to precisely one of 0, bc, $2bc$, ... , $(a - 1)bc$.]

- By Sylvester's theorem, the largest integer that cannot be written in the form $pb+qc$ is $bc-b-c$. Thus, multiplying by a shows that any multiple of a exceeding $a(bc - b - c) = abc - ab - ac$ can be written in the form $pab + qac$.
- But since bc is relatively prime to a, we see that the a integers 0, bc, ... , $(a - 1)bc$ represent all of the residue classes modulo a.
- Therefore, any integer N exceeding $a(bc - b - c) + (a - 1)bc = 2abc - ab - bc - ca$ is congruent to one of those a integers, and the difference $N - kbc$ is a multiple of a larger than $a(bc - b - c)$, hence is of the form $pab+qac$ for some nonnegative p, q by the above. This means $N = kbc+pab+qac$ for nonnegative k, p, q as required.
- However, if $N = 2ab-ab-bc-ca$ could be written as $xbc+yca+zab$ then modulo a we have $-bc \equiv xbc \pmod{a}$ so that $x \equiv -1 \pmod{a}$ and thus $x \geq a - 1$. Likewise we have $y \geq b - 1$ and $z \geq c - 1$. But then $xbc + ycz + zab \geq 3abc - ab - bc - ca > N$, contradiction.

Remark: This is a Frobenius coin problem for the integers ab, bc, ca and was problem 3 from the 1983 IMO.
