E. Dummit's Math 4527 ~ Number Theory 2, Spring 2025 ~ Homework 12, due Tue Apr 15th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the number of Dirichlet characters (a) modulo 5, and (b) modulo 8, and compute their values explicitly.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

- 2. The Carmichael A-function  $\Lambda(n)$  is defined to be  $\ln(p)$  if  $n = p^k$  is a prime power and 0 otherwise. It is frequently used in proofs of the prime number theorem.
  - (a) Show that  $\sum_{d|n} \Lambda(d) = \ln n$ .
  - (b) Show that the Dirichlet series for  $\Lambda$  is  $D_{\Lambda}(s) = -\zeta'(s)/\zeta(s)$  for  $\operatorname{Re}(s) > 1$ .
  - (c) Show that  $D_{\Lambda}(s) = \sum_{p \text{ prime}} \frac{\ln p}{p^s 1}$  for  $\operatorname{Re}(s) > 1$ . [Hint: Use (b) and logarithmic differentiation.]
- 3. If S is a set of positive integers, its <u>natural density</u> is defined to be the value  $\delta(S) = \lim_{N \to \infty} \frac{\#[S \cap \{1, 2, 3, \dots, N\}]}{N}$  if the limit exists.
  - (a) Show that the natural density of the set of even integers is equal to 1/2.
  - (b) Show that the natural density of the set of perfect squares is equal to 0.
  - (c) Show that the natural density of any finite set of positive integers is equal to 0.
  - (d) Show that the natural density of the set of integers with leading digit 1 (in base 10) is undefined. [Hint: Show that the proportion of such integers is at least 50% at  $N = 2 \cdot 10^d$  and at most 20% at  $N = 10^{d+1}$ , for any d.]
  - **Remark:** Note that this problem's version of natural density is not exactly the one we use in class, since the one used in class was only for sets of primes, relative to the set of all primes. (This problem is posed for all integers since it is easier to use this notion of natural density.)
- 4. Let G be a finite abelian group with dual group  $\hat{G}$ , and recall the inner products  $\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$ on functions  $f: G \to \mathbb{C}$  and  $\langle \hat{f}_1, \hat{f}_2 \rangle_{\hat{G}} = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}_1(\chi) \overline{\hat{f}_2(\chi)}$  on functions  $\hat{f}: \hat{G} \to \mathbb{C}$ . Also recall the Fourier transform of a function  $f: G \to \mathbb{C}$  is the function  $\hat{f}: \hat{G} \to \mathbb{C}$  with  $\hat{f}(\chi) = \langle f, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\chi(g)}$ , and the Fourier inversion formula  $f(g) = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g)$  for each  $g \in G$ .
  - (a) Prove Plancherel's theorem:  $\frac{1}{|G|} \langle f_1, f_2 \rangle_G = \left\langle \hat{f}_1, \hat{f}_2 \right\rangle_{\hat{G}}$  for any functions  $f_1, f_2 : G \to \mathbb{C}$ . [Hint: Write  $\hat{f}_1(\chi)$  as a sum over  $g \in G$  and  $\hat{f}_2(\chi)$  as a sum over  $h \in G$ , then use the fact that  $\sum_{\chi \in \hat{G}} \overline{\chi(g)}\chi(h)$  is either |G| or 0 according to whether g = h or not.]
  - (b) Deduce Parseval's theorem:  $\frac{1}{|G|}\sum_{g\in G}|f(g)|^2=\sum_{\chi\in \hat{G}}\left|\hat{f}(\chi)\right|^2.$

- 5. The goal of this problem is to give some approximate prime-number heuristics. We start with the prime number theorem: that there are approximately  $\frac{X}{\ln X}$  primes less than X. Let us make a heuristic assumption (contrary to actual reality) that there are in fact exactly  $\frac{X}{\ln X}$  primes less than X, and also that the primes are otherwise randomly distributed, so that the "probability" that X is prime equals  $\frac{1}{\ln X}$ .
  - (a) Find the number of primes less than or equal to  $X \ln X$ . Use this to explain why the Xth prime is approximately  $X \ln X$ .
  - (b) Under the heuristic from (a), estimate the divergence rate of the sum f(X) =the sum of the reciprocals of the first X primes, by comparing it to an integral.
  - (c) Under the heuristic, explain why there are approximately  $\frac{X}{(\ln X)^2}$  twin primes (primes p such that p+2 is also prime) less than X. [Hint: What is the "probability" that both X and X+2 are prime?]
  - (d) Using the heuristic from (c), estimate the value of the Xth twin prime and then use this to explain why the sum  $\sum_{p \text{ twin prime} \le X} 1/p$  should converge to a finite value.
  - (e) Prove that there are infinitely many primes p that are not twin primes. [Hint: Apply Dirichlet.]
  - **Remark:** Somewhat famously, it is still an open problem whether there exist infinitely many twin primes, so we do not know whether the heuristic from (c) is actually correct. However, it was shown in 1915 by Brun that there exists an upper bound of the form  $CX/(\ln X)^2$  for some C > 0, which by the same kind of argument as in (d) is good enough to prove that the sum of the reciprocals of the twin primes is finite. It was conjectured in 1923 by Hardy and Littlewood that the asymptotic constant C is equal to  $2\prod_{p>3}(1-(p-1)^{-2})$ .
- 6. The goal of this problem is to evaluate some Dirichlet L-series at 1.
  - (a) Let  $\chi_4$  be the nontrivial Dirichlet character mod 4. Show  $L(1,\chi_4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 \frac{1}{3} + \frac{1}{5} \frac{1}{7} + \cdots$
  - (b) Let  $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$  for |x| < 1. Show that  $F'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$  and deduce that  $L(1,\chi_4) = F(1) = \int_0^1 \frac{1}{1+x^2} dx = \pi/4$ . [Hint: Since the series for F converges absolutely, it can be differentiated term by term.]
  - (c) Let  $\chi_3$  be the nontrivial Dirichlet character modulo 3. Show that  $L(1,\chi_3) = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$ .
  - (d) Let  $G(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+1)(3n+2)}$  for |x| < 1. Show that  $G(1) = \int_0^1 \int_0^y \frac{1}{1-x^3} dx dy$  and use this to compute the value of  $L(1,\chi_3)$ . [Hint: Note that  $G''(x) = (1-x^3)^{-1}$  for |x| < 1. For the integral, change the order of integration.]

- 7. [Challenge] Let p be a prime and let  $\chi$  be the Legendre symbol modulo p. The goal of this problem is to evaluate  $L(1,\chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k}$  explicitly, thus generalizing the calculations in problem 6, and then to use this evaluation to prove a formula for the class number in terms of the number of quadratic residues and nonresidues on the interval [1, (p-1)/2] when  $p \equiv 3 \pmod{4}$ . Recall the Gauss sum  $g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^n$  where  $\zeta = e^{2\pi i/p}$  is a primitive pth root of unity, and the general Gauss sum  $g_k(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^{kn}$ .
  - (a) Show that  $-\log(1-\zeta^n) = \sum_{k=1}^{\infty} \frac{\zeta^{nk}}{k}$ . (Note that this series only converges conditionally.)
  - (b) Let  $S = \sum_{n=1}^{p-1} \chi(n) \cdot [-\log(1-\zeta^n)]$ . Prove that  $S = g(\chi)L(1,\chi)$ . [Hint: Use (a), switch summation order, and use the Gauss sum identity  $g_k(\chi) = \chi(k)^{-1}g(\chi)$ .]
  - (c) Define  $P = \frac{\prod_{n \in NR} (1 \zeta^n)}{\prod_{n \in QR} (1 \zeta^n)}$  where NR is the set of quadratic nonresidues modulo p and QR is the set of quadratic residues modulo p. Show that  $P = \exp(g(\chi)L(1,\chi))$ .
  - (d) Find the value of  $L(1, \chi)$  for the Legendre symbol modulo 3. [Hint: The result of (c) is easier to calculate with, unless you like complex logarithms.]
  - (e) Show that if  $p \equiv 3 \pmod{4}$ , so that  $\chi(-1) = -1$ , then  $S = -\frac{i\pi}{p} \sum_{n=1}^{p-1} \chi(n) \cdot n$  where S is as defined in (b). [Hint: In (b), interchange n with -n and add the two sums together.]
  - (f) Show that when  $p \equiv 3 \pmod{4}$  and p > 3 we have  $h(-p) = -\frac{1}{p} \sum_{n=1}^{p-1} \chi(n) \cdot n$ . [Hint: Use the Gauss sum evaluation  $g(\chi) = i\sqrt{p}$  and the analytic class number formula.]
  - (g) Show that when  $p \equiv 3 \pmod{4}$  and p > 3 we have  $h(-p) = \frac{1}{2-\chi(2)} \sum_{n=1}^{(p-1)/2} \chi(n)$ . [Hint: Decompose  $\sum_{n=1}^{p-1} \chi(n) \cdot n$  into two ranges in two different ways: one into even and odd, and another into [1, (p-1)/2] and p [1, (p-1)/2].]
  - (h) Deduce that when  $p \equiv 3 \pmod{4}$ , the class number of  $\mathcal{O}_{\sqrt{-p}}$  is equal to  $\frac{1}{2-\chi(2)}$  times the number of quadratic residues in [1, (p-1)/2] minus the number of quadratic nonresidues on that interval, so in particular there are always more quadratic residues than quadratic nonresidues. Also deduce in particular that this class number is always odd.
  - (i) Find the class numbers of  $\mathcal{O}_{\sqrt{-7}}$ ,  $\mathcal{O}_{\sqrt{-11}}$ ,  $\mathcal{O}_{\sqrt{-19}}$ , and  $\mathcal{O}_{\sqrt{-31}}$ . [If you're still here at this point, for convenience  $\chi(2) = 1$  when  $p \equiv 7 \pmod{8}$  and  $\chi(2) = -1$  when  $p \equiv 3 \pmod{8}$ .]