Dynamics, Chaos, and Fractals (part 5): Introduction to Complex Dynamics (by Evan Dummit, 2025, v. 2.50)

Contents

5 Introduction to Complex Dynamics

In this chapter, our goal is to discuss complex dynamical systems in the complex plane. As we will discuss, much of the general theory of real-valued dynamical systems will carry over to the complex setting, but there are also a number of new behaviors that will occur.

We begin with a brief review of the complex numbers $\mathbb{C}$ and the complex derivative of a complex-valued function, and then turn our attention to the basic study of complex dynamical systems: fixed points and periodic points, attracting and repelling fixed points and cycles, behaviors of neutral fixed points, attracting basins, and orbit analysis. (We will invoke some general results from complex analysis as needed without proof.)

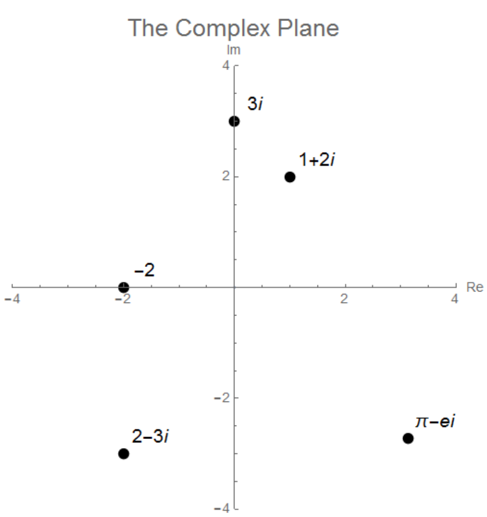
We will then study the structure of Julia sets, which are (roughly speaking) the sets on which a complex function behaves chaotically, and which are often fractals. We close with a study of the famed Mandelbrot set, which is another fractal-like set that arises in the study of the quadratic family $q_c(z) = z^2 + c$. Almost all major theorems in these later sections require significantly more theoretical development in order to provide proofs, so our discussion will primarily be a survey.

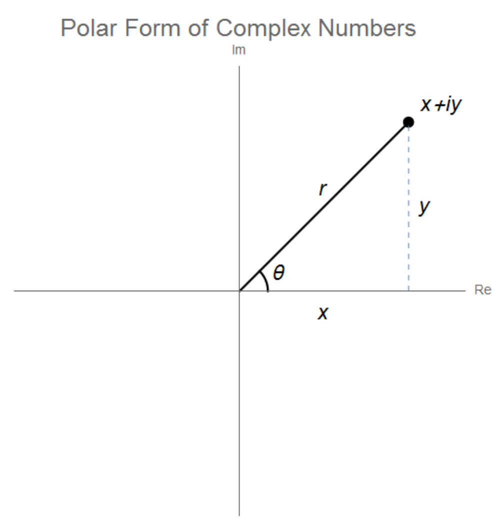
5.1 Dynamical Properties of Complex-Valued Functions

- In this section we review complex arithmetic and complex derivatives, and then apply them to analyze dynamical properties of complex-valued functions, such as attracting and repelling behavior of cycles. Many of our results will parallel quite closely our analysis of the dynamics of real-valued functions.

5.1.1 Complex Arithmetic

- Definitions: A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the so-called "imaginary unit", defined so that $i^2 = -1$. (Frequently, i is written as $\sqrt{-1}$.) The set of all complex numbers is denoted $\mathbb{C}$.
  - The real part of $z = a + bi$, denoted $\text{Re}(z)$, is the real number a, while the imaginary part of $z = a + bi$, denoted $\text{Im}(z)$, is the real number b. The complex conjugate of $z = a + bi$, denoted $\overline{z}$, is the complex number $a - bi$.
  - The modulus (also absolute value, magnitude, or length) of $z = a + bi$, denoted $|z|$, is the real number $\sqrt{a^2 + b^2}$.
  - Two complex numbers are added (or subtracted) simply by adding (or subtracting) their real and imaginary parts: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
  - Two complex numbers are multiplied using the distributive law and the fact that $i^2 = -1$: $(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.
  - For division, we rationalize the denominator: $\dfrac{a + bi}{c + di} = \dfrac{(a + bi)(c - di)}{(c + di)(c - di)} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}\, i$.
  - Example: If $z = 3 + 2i$ and $w = 2 - 2i$, then $z + 2w = 7 - 2i$, $zw = 10 - 2i$, and $\dfrac{z}{w} = \dfrac{1 + 5i}{4}$.
- Here are a few more simple properties of complex number arithmetic:
- Proposition (Complex Arithmetic): Suppose z and w are complex numbers.
  1. We have $\text{Re}(z) = (z + \overline{z})/2$ and $\text{Im}(z) = (z - \overline{z})/(2i)$.
  2. We have $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$, and $\overline{\overline{z}} = z$.
  3. We have $|\overline{z}| = |z|$ and $|zw| = |z| \cdot |w|$.
  4. We have $z = \overline{z}$ if and only if z is real, while $\overline{z} = -\overline{z}$ if and only if z is purely imaginary (of the form ri where r is real).
  5. We have $\text{Re}(z) \le |z|$ and $\text{Im}(z) \le |z|$.
  6. (Triangle Inequality) We have $|z + w| \le |z| + |w|$.
     - Proofs: (1)-(5) are easy algebraic calculations.
     - For (6), use (1) and (2) to observe $z\overline{w} + w\overline{z} = 2\text{Re}(z\overline{w})$, and (5) and (3) to observe $2\text{Re}(z\overline{w}) \le 2\,|z\overline{w}| = 2\,|z|\,|w|$.
     - Then $|z + w|^2 = (z+w)(\overline{z + w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + |w|^2 + 2\text{Re}(z\overline{w}) \le |z|^2 + |w|^2 + 2\,|z|\,|w| = (|z| + |w|)^2$. Since both $|z + w|$ and $|z| + |w|$ are nonnegative, taking the square root yields the desired $|z + w| \le |z| + |w|$.
- We often think of the real numbers geometrically, as a line. The natural way to think of the complex numbers is as a plane, with the x-coordinate denoting the real part and the y-coordinate denoting the imaginary part.





- Once we do this, there is a natural connection to polar coordinates: namely, if $z = x + yi$ is a complex number which we identify with the point (x, y) in the complex plane, then the modulus $|z| = \sqrt{x^2 + y^2}$ is simply the coordinate r when we convert (x, y) from Cartesian to polar coordinates.
- Furthermore, if we are given that $|z| = r$, we can uniquely identify z given the angle θ that the line connecting z to the origin makes with the positive real axis. (This is the same θ from polar coordinates.)

- From polar coordinates (or simple trigonometry), we see that we can write z in the form $z = r[\cos(\theta) + i\sin(\theta)]$, which is called the polar form of z.
  - The length r is simply the modulus of z, while the angle θ is called the argument of z and sometimes denoted $\theta = \arg(z)$.
  - We will emphasize that although r is unique, θ is not: since the sine and cosine are periodic with period 2π, any θ that differs by an integral multiple of 2π yields an equivalent polar form. We will implicitly identify polar forms yielding the same complex number.
  - Example: For $r = 2$ and $\theta = 0$ we obtain $z = 2[\cos 0 + i\sin 0] = 2$. Taking $r = 2$ and $\theta = 2\pi$ also yields $z = 2[\cos 2\pi + i\sin 2\pi] = 2$.
  - Conversely, if we know $z = x + iy$ then we can compute the (r, θ) form fairly easily by solving $x = r\cos\theta$ and $y = r\sin\theta$ for r and θ.
  - Explicitly, we have $r = \sqrt{x^2 + y^2} = |z|$ and we can take $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ if $x > 0$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ if $x < 0$. (The extra $+\pi$ is needed when $x < 0$ because of the fact that the principal arctangent function only has range $(-\pi/2, \pi/2)$, so we would otherwise get the wrong value for θ if z lies in the second or third quadrants.)
  - Example: If $z = 1 + i$, then the corresponding values of r and θ above are $r = |z| = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$, so we can write z in polar form as $z = \boxed{\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]}$. Indeed, we may check that $\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 1 + i$, as it should be.
  - Example: If $z = -1 + i\sqrt{3}$, then the corresponding values of r and θ above are $r = |z| = 2$ and $\theta = \pi + \tan^{-1}(-\sqrt{3}) = 2\pi/3$, so we can write z in polar form as $z = \boxed{2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]}$.

- We can also repackage the polar form using complex exponentials:
- Definition: If $z = x + iy$ is a complex number, we define the complex exponential $e^z = e^x(\cos y + i\sin y)$.
  - Examples: We have $e^{i\pi/2} = e^0(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = \boxed{i}$ and $e^{1+i\pi} = e^1(\cos\pi + i\sin\pi) = \boxed{-e}$.
  - The motivation here is that we want the complex exponential to obey the familiar rules for the real exponential function, and so in particular we want $e^{x+iy} = e^x e^{iy}$.
  - It therefore suffices just to define e^{iy}, which we do via Euler's identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
  - Euler's identity encodes a lot of information: for example, we claim that $e^{i\theta} \cdot e^{i\varphi} = e^{i(\theta+\varphi)}$. Expanding both sides with Euler's identity yields

$$\begin{aligned} e^{i\theta} \cdot e^{i\varphi} &= [\cos\theta + i\sin\theta][\cos\varphi + i\sin\varphi] \\ &= [\cos\theta\cos\varphi - \sin\theta\sin\varphi] + i[\sin\theta\cos\varphi + \cos\theta\sin\varphi] \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)} \end{aligned}$$

    where the key step in the middle uses the usual addition identities $\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$ and $\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$ for sine and cosine.
  - Another way of interpreting this calculation is that the (initially rather arbitrary-seeming) sine and cosine addition formulas actually just reflect the natural structure of the multiplication of complex numbers.

- Another convenient result follows by applying Euler's identity to the simple relation $e^{i(n\theta)} = (e^{i\theta})^n$, which when written out yields De Moivre's identity $\cos(n\theta) + i\sin(n\theta) = [\cos\theta + i\sin\theta]^n$.

- By expanding the right-hand side using the binomial theorem we can obtain identities for $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin\theta$ and $\cos\theta$.
- Example: Setting $n = 2$ produces $\cos(2\theta) + i\sin(2\theta) = [\cos\theta + i\sin\theta]^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$, and so we recover the double-angle formulas $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
- Example: Setting $n = -1$ produces $\cos(-\theta) + i\sin(-\theta) = [\cos\theta + i\sin\theta]^{-1} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$, and so we recover the standard identities $\cos^2\theta + \sin^2\theta = 1$, $\cos(-\theta) = \cos\theta$, and $\sin(-\theta) = -\sin\theta$.

- Using Euler's identity and the polar form of complex numbers above, we see that every complex number can be written in exponential form as $z = r \cdot e^{i\theta}$ for the same r and θ we described above.
  - Example: We can draw $1 + i$ in the complex plane, or use the formulas, to see that $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \frac{\pi}{4}$, and so we see that $1 + i = \boxed{\sqrt{2} \cdot e^{i\pi/4}}$.
  - Example: Either by geometry or trigonometry, we see that $\left|1 - i\sqrt{3}\right| = 2$ and $\arg(1 - i\sqrt{3}) = -\frac{\pi}{3}$, hence $1 - i\sqrt{3} = \boxed{2 \cdot e^{-i\pi/3}}$.
  - Example: Using the formulas for r and θ above, we have $3 + 2i = \boxed{\sqrt{13} \cdot e^{i \cdot \arctan(2/3)}}$.

- The rectangular $a + bi$ form of a complex number is more convenient for addition, while the polar $re^{i\theta}$ form is more convenient for multiplication, since we may easily multiply $(re^{i\theta})(se^{i\varphi}) = (rs)e^{i(\theta+\varphi)}$. (This calculation is often summarized as "lengths multiply, angles add".)
  - In particular, it is very easy to take powers of complex numbers when they are in exponential form: we have $(r \cdot e^{i\theta})^n = r^n \cdot e^{i(n\theta)}$.

- Example: Compute $(1 + i)^8$.
  - From above we have $1 + i = \sqrt{2} \cdot e^{i\pi/4}$, so $(1 + i)^8 = \left(\sqrt{2} \cdot e^{i\pi/4}\right)^8 = (\sqrt{2})^8 \cdot e^{8i\pi/4} = 2^4 \cdot e^{2i\pi} = \boxed{16}$. (Note how much easier this is compared to multiplying $1 + i$ by itself eight times.)

- Example: Compute $(1 - i\sqrt{3})^9$.
  - From above we have $1 - i\sqrt{3} = 2 \cdot e^{-i\pi/3}$, so $(1 - i\sqrt{3})^9 = 2^9 \cdot e^{-9i\pi/3} = 512 \cdot e^{-3i\pi} = \boxed{-512}$.

- Many textbooks introduce complex numbers as a tool for giving meaning to the formal symbols obtained when using the quadratic formula to "solve" quadratic equations that do not have real solutions.
  - Explicitly, if a, b, c are real numbers and $a \neq 0$, then we may complete the square in the expression $az^2 + bz + c$ and write it as $a(z + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$.
  - We may then obtain the usual quadratic formula for the roots of the polynomial $az^2 + bz + c = 0$; namely, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - In the situation where $b^2 - 4ac < 0$, the solutions are not real numbers but rather complex numbers. For example, it indicates that the solutions to $z^2 + 2z + 2 = 0$ are $z = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$.
  - Indeed, we can check that if we evaluate the expression $z^2 + 2z + 2$ when $z = -1 + i$ or $-1 - i$, we obtain 0.

- More generally, the Fundamental Theorem of Algebra says that any polynomial equation $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ can be completely factored as a product of linear terms over the complex numbers. (This a foundational result in algebra and the first complete and correct proof was given by Argand and Gauss in the early 1800s.)

5.1.2 Limits, Continuity, and Complex Derivatives

- Now that we have discussed the arithmetic of the complex numbers, we begin our study of complex-valued functions $f : \mathbb{C} \to \mathbb{C}$.
- Our first main goal is to generalize the notion of the derivative of a function to the complex case. There is a very natural way to try to do this; namely, by defining the complex derivative of a function as a limit in the same way as with a real-valued function.
  - This is in fact the definition we will take, but since it involves a limit, we briefly mention complex limits.
- Definition: A function $f : \mathbb{C} \to \mathbb{C}$ has the limit L as $z \to a$, written as $\lim\limits_{z\to a} f(z) = L$ if for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) such that for all $z \in \mathbb{C}$ with $0 < |z - a| < \delta$, we have $|f(z) - L| < \epsilon$.
  - This is simply the usual ϵ-δ definition of limit but with the variables having domain $\mathbb{C}$ rather than $\mathbb{R}$.
  - We can mostly avoid using the formal definition, as with real-valued functions, by instead working with various limit rules, such as the fairly obvious $\lim_{z\to a} z = a$ and $\lim_{z\to a} c = c$ for any constant c, along with rules for combining limits: if $\lim\limits_{z\to a} f(z) = L_f$ and $\lim\limits_{z\to a} g(z) = L_g$, then for example we have $\lim_{z\to a} |f(z)| = |L_f|$, $\lim_{z\to a} [f(z) + g(z)] = L_f + L_g$, $\lim_{z\to a} f(z)g(z) = L_f L_g$, and $\lim\limits_{z\to a} f(z)/g(z) = L_f/L_g$when $L_g \neq 0$.
- We also have the natural notion of continuity:
- Definition: If $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued function, we say f is continuous at $a \in \mathbb{C}$ if $\lim_{z\to a} f(z) = f(a)$. If f is continuous on its entire domain, we say f is continuous everywhere (or often, just continuous).
  - In other words, a continuous function is one whose limit as $z \to a$ is simply the value of the function at a.
  - Per the basic limit properties, we see that sums, differences, products, and quotients (with nonzero denominator) of continuous functions are continuous.
  - Example: Since polynomials are constructed from constants and the variable z using addition, subtraction, and multiplication, any polynomial $p(z)$ is continuous everywhere. More generally, any rational function $p(z)/q(z)$ is continuous whenever $q(z) \neq 0$.
- Now we can define the complex derivative, quite analogously to the derivative of a real-valued function:
- Definition: If $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued function, the complex derivative $f'(z_0)$ at a point $z_0 \in \mathbb{C}$ is the limit $\lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, assuming it exists.
  - As with the derivative of a real-valued function, we can equivalently state this definition as $f'(z) = \lim\limits_{h\to 0} \dfrac{f(z+h) - f(z)}{h}$.
  - In some cases the first definition is easier to use, while in others the second definition is easier to use.
- Example: Verify that the complex derivative of $f(z) = z^2$ exists everywhere and compute it.
  - We compute $\lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{z\to z_0} \dfrac{(z - z_0)(z + z_0)}{z - z_0} = \lim\limits_{z\to z_0} (z + z_0) = 2z_0$.
  - Since the limit always exists, we can say that $f'(z_0) = \boxed{2z_0}$ everywhere.
  - Note that this agrees with, and in fact extends, the (ordinary) real-valued derivative of the function $f(x) = x^2$ on the real line.
- Example: Show that the complex derivative of $f(z) = \overline{z}$ does not exist anywhere.
  - The required limit is $\lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{z\to z_0} \dfrac{\overline{z} - \overline{z_0}}{z - z_0}$. There does not seem to be a natural way to simplify this expression further.

- Let us try to compute the limit along different paths approaching z_0.
- Along the horizontal line $z = z_0+t$, for a real parameter $t \to 0$, the limit becomes $\lim_{t\to 0} \dfrac{\overline{(z_0+t)} - \overline{z_0}}{(z_0+t)-z_0} = \lim_{t\to 0} \dfrac{t}{t} = 1$.
- Along the vertical line $z = z_0+it$ for a real parameter $t \to 0$, the limit becomes $\lim_{t\to 0} \dfrac{\overline{(z_0+it)} - \overline{z_0}}{(z_0+it)-z_0} = \lim_{t\to 0} \dfrac{-it}{it} = -1$.
- Along these two paths the limit has different values, so the overall limit does not exist at any point z_0. This means the derivative $\boxed{\text{does not exist}}$ anywhere.

- Since our definition of the complex derivative is exactly the same as that of the derivative of a real-valued function, all of the usual differentiation rules carry over verbatim: the product and quotient rules, the chain rule, the linearity of the derivative, etc.
  - As a consequence of the basic differentiation rules, we can see that any complex polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is differentiable with complex derivative $p'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$.
  - Example: The complex derivative of $p(z) = iz^4 - (3+i)z + 5$ is $p'(z) = 4iz^3 + (3+i)$.
- We can extend this result to differentiate power series, as follows:
- Proposition (Complex Power Series): Suppose $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=0}^{\infty} a_n z^n$ is defined by a complex power series where $R = \lim_{n\to\infty} |a_n|^{-1/n}$ exists. Then the power series converges absolutely[1] for all z with $|z| < R$ and diverges for all z with $|z| > R$. On the region $|z| < R$, the power series is differentiable with complex derivative obtained by differentiating term-by-term: $f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=1}^{\infty} na_n z^{n-1}$.
  - The value R is called the radius of convergence of the power series, because the series converges inside the circle $|z| = R$ and diverges outside it. More generally, if we have *any* complex power series $\sum_{n=0}^{\infty} a_n z^n$, its radius of convergence can be shown to equal $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$.
  - Proof (outline): The convergence behavior of the series follows by noting that $|a_n| \sim 1/R^n$ for large n, so when $|z| < R$ the series $\sum_{n=0}^{\infty} |a_n z^n|$ is bounded above by the convergent geometric series $\sum_{n=0}^{\infty} |z/R|^n$, and when $|z| > R$ the individual terms $|a_n z^n| \sim |z/R|^n$ do not tend to zero so the series diverges. (Making the estimates rigorous requires some minor additional care.)
  - The differentiability of the series follows by a direct calculation: $f'(z) = \lim_{h\to 0} \dfrac{1}{h} \sum_{n=0}^{\infty} a_n[(z+h)^n - z^n] = \lim_{h\to 0} \sum_{n=0}^{\infty} a_n[nz^{n-1} + \binom{n}{2} hz^{n-2} + \cdots + h^{n-1}] = \sum_{n=0}^{\infty} a_n z^{n-1}$. To justify interchanging the limit and infinite sum requires various nontrivial results about uniform convergence of sequences of functions (or equivalently, making an estimate on the difference and showing it does have limit zero), whose details we omit.
- As a very pleasant consequence of these results, because we can differentiate complex power series term by term inside their radius of convergence just as we differentiate real power series term by term, and because all of the usual differentiation rules hold as well, complex derivatives of familiar real-valued functions will have exactly the same formulas as their real counterparts.
  - Example: The complex derivative of $f(z) = \dfrac{1}{1+z}$ is $f'(z) = -\dfrac{1}{(1+z)^2}$ by the chain and power rules.
  - Example: For $f(z) = e^z = 1 + z + z^2/2 + z^3/3! + \cdots = \sum_{n=0}^{\infty} z^n/n!$, we have $f'(z) = 1 + z + z^2/2 + \cdots = \sum_{n=1}^{\infty} nz^{n-1}/n! = \sum_{k=0}^{\infty} z^k/k! = e^z$, just as we would expect.
  - In particular, rational functions of z are complex-differentiable on large regions (namely, their entire domains, which consist of the entire complex plane except the finite number of points where their denominators are zero), and functions defined by power series will likewise be complex-differentiable on their disc of convergence.
  - Since our primary objects of study will be functions that are complex-differentiable on such large regions, we give them a simpler name.

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[1] A complex series $\sum_{n=0}^{\infty} b_n$ converges when its the limit of its sequence of partial sums $\lim_{k\to\infty} \sum_{n=0}^{k} b_n$ converges, and the series converges absolutely when its absolute value series $\sum_{n=0}^{\infty} |b_n|$ converges.

- Definition: A function whose complex derivative exists on a nonempty open region U is said to be holomorphic on U.
  - Although being holomorphic seems to be a relatively mild condition, it actually turns out to be quite restrictive.
  - For example, even though by definition a holomorphic function only possesses a first derivative, in fact a holomorphic function necessarily has derivatives of *all* orders (compare with the situation with real-differentiable functions, which may not even have a second derivative).
  - Furthermore, as nearly an immediate consequence of having derivatives of all orders, holomorphic functions may be represented locally by power series with a positive radius of convergence on which they converge to the value of the original function. (Thus, in fact our statement about differentiability of power series actually applies to any holomorphic function, since they can all be expressed as convergent power series.)

5.1.3 Attracting, Repelling, and Neutral Fixed Points and Cycles

- Now that we have a notion of derivative for complex-valued functions, our next task is to define the notion of an attracting, repelling, or neutral fixed point (or cycle) in terms of the value of the complex derivative.
  - In general, a polynomial function $p(z)$ of degree d will usually have d^n points of period dividing n: the equation $p^n(z) - z = 0$ is an equation of degree d^n, which, by the fundamental theorem of algebra, will have exactly d^n roots (counted with multiplicity) in $\mathbb{C}$.
  - Unless something unusual happens (i.e., unless many of the roots occur with high multiplicity), a polynomial will generally have a point of exact period n for each n: indeed, it is a theorem of I.N. Baker that if $f(z)$ is a polynomial of degree $d > 1$ that is not conjugate to the polynomial $z^2 - 3/4$ by a linear change of variables, then $f(z)$ has a point of exact period n for every $n \geq 1$.
- Definition: If z_0 is a fixed point of the holomorphic function f, we say z_0 is an attracting fixed point if $|f'(z_0)| < 1$, we say z_0 is a repelling fixed point if $|f'(z_0)| > 1$, and we say z_0 is a neutral fixed point if $|f'(z_0)| = 1$.
- Definition: We say that a periodic point x_0 for f is attracting (respectively, repelling or neutral) if x_0 is an attracting (respectively, repelling or neutral) fixed point for f^n.
  - The key results, as with the real case, is that an attracting fixed point (or cycle) will attract nearby orbits, and a repelling fixed point (or cycle) will repel nearby orbits.
- Theorem (Attracting Points): If z_0 is an attracting fixed point of the holomorphic function f, then there exists an open disc D of positive radius centered at z_0 such that, for any $z \in D$, $f^n(z) \in D$ for all $n \geq 1$. Furthermore, for any $z \in D$, it is true that $f^n(z) \to z_0$ as $n \to \infty$, and the convergence is exponentially fast.
  - The proof is essentially the same as in the real case.
  - Proof: By definition, $|f'(z_0)| = \lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < 1$.
  - Since $\frac{f(z) - f(z_0)}{z - z_0}$ is a continuous function of z with a removable discontinuity at $z = z_0$, there exists a constant $\lambda < 1$ and a positive r such that for all z with $|z - z_0| < r$, it is true that $\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \lambda$.
  - Rearranging yields $|f(z) - z_0| < \lambda \, |z - z_0|$.
  - In particular, we see that $|f(z) - z_0| < \lambda r < r$, so $f(z)$ also lies in the disc $D = \{z : |z - z_0| < r\}$.
  - Iterating the argument yields $f^n(z) \in D$ for all $n \geq 1$. We then conclude that $|f^n(z) - z_0| < \lambda^n \, |z - z_0|$ for all $n \geq 1$.
  - Since $\lambda < 1$, the right-hand term goes to zero as $n \to \infty$, so $f^n(z) \to z_0$.

- Remark: In fact, we can take D to be any disc centered at z_0 such that $|f'(z)| < \lambda$ for any fixed $\lambda < 1$. To see this, if C is the line segment from z_0 to z, the (complex) fundamental theorem of calculus says that $\int_C f'(z)\,dz = f(z) - f(z_0)$. By the triangle inequality for integrals we have $\left|\int_C f'(z)\,dz\right| \le \int_C |f'(z)|\,dz < \int_C \lambda\,dz = \lambda\,|z - z_0|$ since the integral of 1 along a curve is its arclength. Thus we see $|f(z) - f(z_0)| < \lambda\,|z - z_0|$, and the rest of the argument follows as above.

- We also have an analogous result for repelling points:
- Theorem (Repelling Points): If z_0 is a repelling fixed point of the holomorphic function f, then there exists an open disc D of positive radius centered at z_0 such that, for any $z \in D$ with $z \neq z_0$, there exists a positive integer n such that $f^n(z) \notin D$.
  - As before, the proof is essentially identical to that given for the real case.
  - Proof: By the same argument as for the theorem on attracting points, there exists a constant $\lambda < 1$ and a positive r such that for all z with $|z - z_0| < r$, it is true that $\left|\dfrac{f(z) - f(z_0)}{z - z_0}\right| < \lambda$.
  - Thus, $|f(z) - z_0| > \lambda\,|z - z_0|$, and then by a trivial induction we see that $|f^n(z) - z_0| > \lambda^n\,|z - z_0|$, assuming that $f^{n-1}(z)$ lies in $D = \{z : |z - z_0| < r\}$.
  - If the orbit of z never left D, then we would have $|f^n(z) - z_0| > \lambda^n\,|z - z_0|$: but since $\lambda > 1$, the right-hand side tends to infinity as $n \to \infty$. But $f^n(x)$ is assumed to lie in D for all n, meaning that D contains points arbitrarily far away from z_0: contradiction.

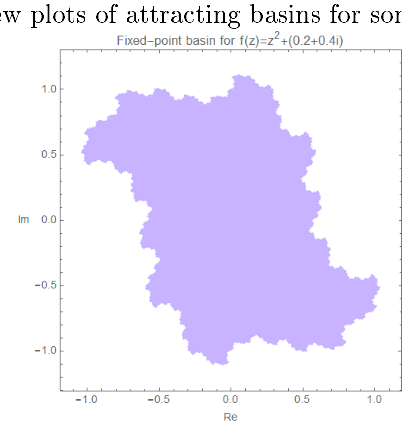
- The next natural step would seem to be to give a classification theorem for neutral fixed points. However, it turns out that neutral fixed points for complex functions can be much more complicated than for real-valued functions, so we will pause that discussion for now.
- Example: Find and classify the fixed points of $f(z) = z^2 - z + 2$ as attracting, repelling, or neutral.
  - The fixed points satisfy $z^2 - 2z + 2 = 0$, whose solutions are $\dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$.
  - Since $f'(z) = 2z - 1$, we see that $f'(1+i) = 1+2i$, so since $|1 + 2i| = \sqrt{5} > 1$ we see that $\boxed{1 + i \text{ is repelling}}$.
  - Similarly, $|f'(1 - i)| = |1 - 2i| = \sqrt{5} > 1$, so $\boxed{1 - i \text{ is repelling}}$ as well.
- Example: Find and classify the fixed points of $g(z) = z^3 - 2iz^2$ as attracting, repelling, or neutral.
  - The fixed points of $g(z)$ are the solutions to $z^3 - 2iz^2 - z = 0$, which factors as $z(z - i)^2 = 0$.
  - Thus, the fixed points are $z = 0$ and $z = i$.
  - Since $g'(z) = 3z^2 - 4iz$, we see that $|g'(0)| = 0$ so $\boxed{0 \text{ is attracting}}$.
  - Similarly, $|g'(i)| = |1| = 1$ so $\boxed{i \text{ is neutral}}$.
- For cycles, we can apply the chain rule to classify the attracting/repelling behavior in the same way as for real-valued functions.
  - Explicitly, if $\{z_1, z_2, \cdots, z_n\}$ is an n-cycle, then it is attracting, neutral, or repelling precisely when the value $P = (f^n)'(z_1) = \prod_{i=1}^{n} |f'(z_i)|$ is respectively less than 1, equal to 1, or greater than 1.
- Example: Classify the periodic cycle containing 0 for $f(z) = z^3 - \dfrac{1+i}{2}(z^2 + z) + i$ as attracting, repelling, or neutral.
  - We have $f(0) = i$, $f(i) = i^3 - \dfrac{1+i}{2}(-1+i) + i = 1$, and $f(1) = 1 - (1+i) + i = 0$, so $\{0, i, 1\}$ is a 3-cycle.
  - Since $f'(z) = 3z^2 - \dfrac{1+i}{2}(2z + 1)$ we can compute $f'(0) = -\dfrac{1+i}{2}$, $f'(i) = -\dfrac{5+3i}{2}$, and $f'(1) = \dfrac{3-3i}{2}$.
  - Then some additional arithmetic eventually yields $f'(0) \cdot f'(i) \cdot f'(1) = \dfrac{15+9i}{4}$, whose absolute value is clearly larger than 1. Thus, the cycle is $\boxed{\text{repelling}}$.

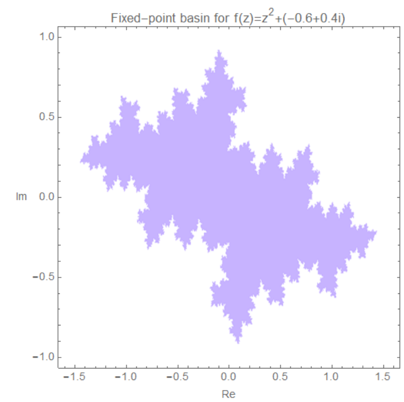
- As with real-valued functions we can also define the attracting basin of an attracting fixed point:

- Definition: If z_0 is an attracting fixed point of a (holomorphic) function f, then the basin of attraction (or attracting basin) for z_0 is the set of all points z such that $f^n(z) \to z_0$ as $n \to \infty$.

  - As in the real case, we can compute the basin of attraction as a union of inverse images: if z_0 is an attracting fixed point of the holomorphic function f and D is any open set containing x_0 that lies in the basin of attraction, then the full basin of attraction B_{z_0} is given by $B_{z_0} = \bigcup_{n=0}^{\infty} f^{-n}(D) = D \cup f^{-1}(D) \cup f^{-2}(D) \cup \cdots$.
  - Unlike in the real case, however, it is much more difficult to compute inverse images of regions in the plane. Holomorphic functions map boundaries to boundaries, so to find general preimages, one may more simply compute the entire inverse image of the boundary of the region. This is not so difficult when the region is an interval (since its boundary consists of the two endpoints) but is very hard even when the region is a circle.
  - In practice, to draw the attracting basin, it is generally faster to compute a large number of iterates of individual points to determine their eventual orbit behaviors than to attempt to compute the basin using inverse images.
  - It is also more difficult to decide what the "immediate" attracting basin is: ultimately, the right answer is that it is the largest connected open set containing z_0 that lies in the attracting basin.
  - We will return shortly to the question of computing attracting basins, which for many functions turn out to be closely related to Julia sets.

- Here are a few plots of attracting basins for some simple functions:





5.1.4 Orbits of Linear Maps on $\mathbb{C}$

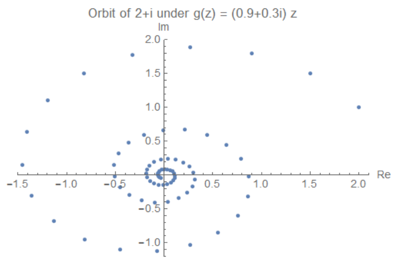
- Let us analyze the orbits of the linear maps $p(z) = az + b$ in terms of a and b.

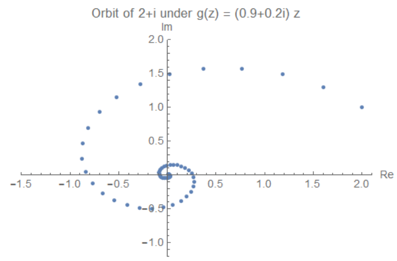
  - If $a = 1$, then the map is simply a shift map $p(z) = z + b$, whose orbits are obvious.
  - So now assume $a \neq 1$. We can see that $p(z)$ has a unique fixed point $z_0 = b/(1 - a)$.
  - It is easy to check that the dynamical system $(\mathbb{C}, p)$ is conjugate via the homeomorphism $h(z) = z - b/(1 - a)$ to the dynamical system $(\mathbb{C}, g)$, where $g(z) = az$, whose fixed point is now at $z = 0$.
  - So we may equivalently analyze the behavior of the simpler "scaling map" where $b = 0$.

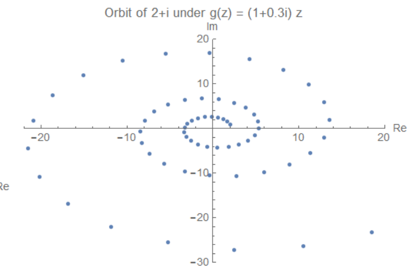
- Let $g(z) = az$ and write $a = re^{i\theta}$.

  - Since $g^n(z) = a^n z = r^n e^{ni\theta} z$ we see that if $r < 1$ then the attracting fixed point $z = 0$ attracts all orbits, and if $r > 1$ then the orbit of every point except the fixed point "goes to ∞" in the sense that $|g^n(z)| \to \infty$.
  - Geometrically, multiplication by $a^n = r^n e^{ni\theta}$ will scale z by a factor of r^n and rotate it by an angle θ around the origin. Thus, for $\theta \neq 0$, the orbits in general will have a "spiral" pattern toward the origin if

$r < 1$ and away from the origin if $r > 1$:







- More interesting is the case $r = 1$, with $a = e^{i\theta}$. In this case, the origin is a neutral fixed point.
  - Since $|g(z)| = |z|$ we see that the orbit of each point moves around a circle centered at the origin. If we restrict our attention to this circle, then $g(z)$ is simply the map that rotates the circle counterclockwise by θ radians.
  - If θ is a rational multiple of 2π (the "rational rotation" case), say $\theta = \dfrac{2\pi p}{q}$ where $\dfrac{p}{q}$ is in lowest terms, then a is a qth root of unity (i.e., $a^q = 1$) and $g^q(z) = z$ for every value of z. In this case, every point is a periodic point of exact period q, except for $z = 0$ which is a fixed point.
  - If θ is an irrational multiple of 2π (the "irrational rotation" case), then a^n will never be equal to 1 for any n, so in this case that the orbit of z will roam around the circle of radius $|z|$ without ever falling into a periodic cycle.
  - Notice in particular that the orbit behavior when 0 is a neutral fixed point can be very complicated and depends very finely on the precise nature of the value of the derivative $g'(0)$. This behavior is in stark contrast to the real case, where there are only two possible types of neutral fixed point.
- In fact, we can say more in the irrational rotation case: it turns out that the orbit is actually dense on the circle of radius $|z|$. Explicitly:
- Theorem (Jacobi): The orbit of any point $z \in \mathbb{C}$ under the map $g(z) = e^{i\theta} z$ where θ is an irrational multiple of 2π is dense on the circle of radius $|z|$.
  - In fact the orbits are actually "asymptotically equally distributed" along the circle, in the sense that the proportion of the first n orbits that land in any given arc of angle ϵ tends to $\dfrac{\epsilon}{2\pi}$ as $n \to \infty$. (We will not prove this result, but it is a special case of a much more general class of results known as equidistribution theorems.)
  - Proof: It is sufficient to show that the orbit of any starting point z_0 will enter any given arc of angle ϵ on the circle of radius $|z_0|$.
  - To do this, choose k such that $2\pi/k < \epsilon$, and consider the points z_0, $g(z_0)$, ... , $g^k(z_0)$ on the circle of radius z_0, which (by hypothesis) are all distinct.
  - Two of these points must have a (counterclockwise) angle of less than $2\pi/k < \epsilon$ between them, since there are $k + 1$ points subtending a total angle of 2π.
  - Suppose these points are $g^i(z_0)$ and $g^j(z_0)$ where $i < j$: then g^{j-i} corresponds to a rotation by an angle less than ϵ.
  - Therefore, the points $g^{n(j-i)}(z)$ for $n \geq 0$ are spaced around the circle at successive angles less than ϵ, so in particular at least one such point must lie inside any given arc of angle ϵ, as required.

5.1.5 Some Properties of Orbits of Quadratic Maps on $\mathbb{C}$

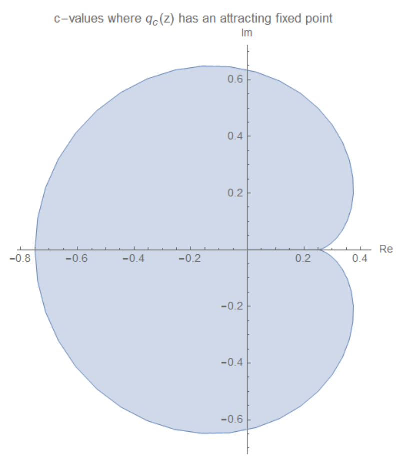
- We now consider the orbits of quadratic maps of the form $p(z) = a_1 z^2 + a_2 z + a_3$ on $\mathbb{C}$.
  - Like in the real case, by applying an appropriate conjugation it is sufficient to consider maps of the form $q_c(z) = z^2 + c$, where now c is an arbitrary complex parameter.

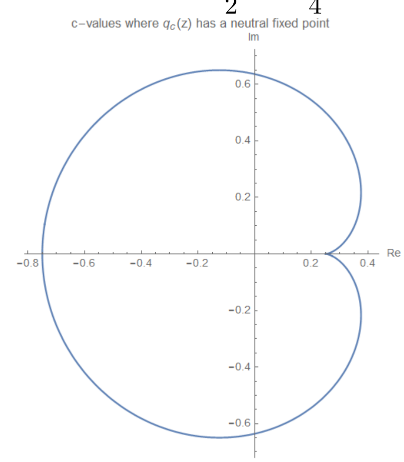
- Indeed, we can use the same conjugation as in the real case, namely, $h(x) = ax + b$ where $a = a_1$ and $b = a_2/2$. It is then straightforward to verify that $h(p(x)) = q_c(h(x))$, where $c = a_1a_3 + a_2/2 - a_2^2/4$.

- So let $q_c(z) = z^2 + c$, and observe that $q_c'(z) = 2z$.

- Our first goal is to characterize the values of c for which $q_c(z)$ has an attracting or neutral fixed point.

  - To do this, suppose one fixed point is $z_0 = re^{i\theta}$.
  - Then, since $f'(z) = 2z$, we see that z_0 will be attracting for $0 \le r < \frac{1}{2}$, neutral when $r = \frac{1}{2}$, and repelling for $r > \frac{1}{2}$.
  - We have $c = z_0 - z_0^2 = re^{i\theta} - r^2e^{2i\theta}$, and by factoring $z^2 - z + c = 0$ we see that the other root is $z_1 = 1 - re^{i\theta}$.
  - By the triangle inequality, if $r < \frac{1}{2}$, then $|z_1| \ge 1 - r > \frac{1}{2}$, so if z_0 is attracting then z_1 is necessarily repelling.
  - Furthermore, if $r = \frac{1}{2}$, then $|z_1| \ge 1 - r \ge \frac{1}{2}$ with equality if and only if $e^{i\theta} = 1$: namely, when $z_0 = z_1 = \frac{1}{2}$, with $c = \frac{1}{4}$. For $c \ne 1/4$, if z_0 is neutral then z_1 is also necessarily repelling.
  - From this we conclude that, for $c \ne 1/4$ (so that there are two distinct fixed points), one of them is always repelling, and the other can be attracting, repelling, or neutral.
  - If we plot the values of c where $0 \le r < \frac{1}{2}$ in the complex plane for $0 \le \theta \le 2\pi$, we obtain the open interior of the cardioid whose boundary is the set of points of the form $c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$:



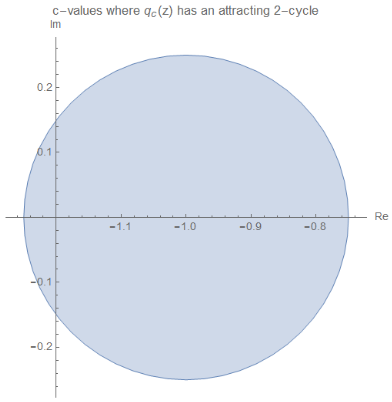


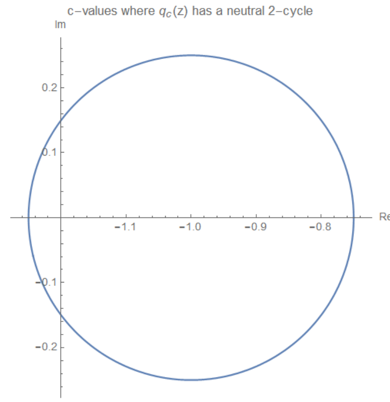
  - On the interior of the cardioid at the point $c = re^{i\theta} - r^2e^{2i\theta}$ for $0 \le r < \frac{1}{2}$, the fixed point $z_0 = re^{i\theta}$ is attracting and the other fixed point $z_1 = 1 - re^{i\theta}$ is repelling.
  - On the boundary of the cardioid at the point $c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$ for $0 < \theta < 2\pi$, the fixed point $z_0 = \frac{1}{2}e^{i\theta}$ is neutral and the other fixed point $z_1 = 1 - \frac{1}{2}e^{i\theta}$ is repelling.
  - Outside the cardioid, both fixed points are repelling.

- In a similar way, we can analyze the behavior of the 2-cycle for $q_c(z)$.

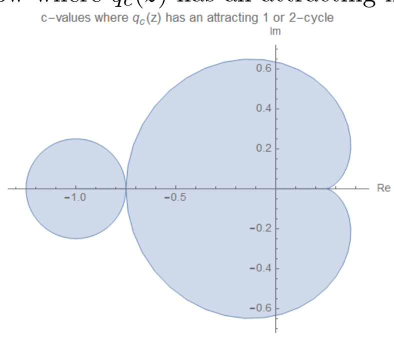
  - The two period-2 points z_0 and z_1 are the roots of $\frac{q_c^2(z) - z}{q_c(z) - z} = z^2 + z + (c+1)$. From the factorization we see that $z^2 + z + (c+1) = (z - z_0)(z - z_1)$, and thus $z_0z_1 = c + 1$.
  - By the chain rule, we see that $(q_c^2)'(z_0) = q_c(z_0)q_c(z_1) = 4z_0z_1 = 4(c+1)$.

- Therefore, the 2-cycle is attracting precisely when $|c+1| < \frac{1}{4}$, neutral when $|c+1| = \frac{1}{4}$, and repelling when $|c+1| > \frac{1}{4}$, which is a circle in the plane:





- We can overlay these two plots to show where $q_c(z)$ has an attracting fixed point or 2-cycle:



  - Note that the cardioid is tangent to the circle at $c = -3/4$, which is the location of the period-doubling bifurcation of the quadratic family.
  - We could continue this procedure, and also plot the regions for which $q_c(z)$ has an attracting 3-cycle, 4-cycle, 5-cycle, and so forth: this will ultimately produce the famous Mandelbrot set. (We will discuss it in more depth very soon.)

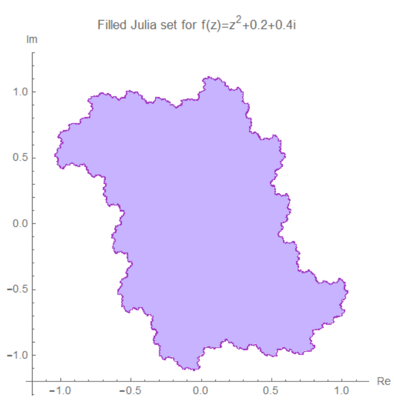
5.2 Julia Sets For Polynomials

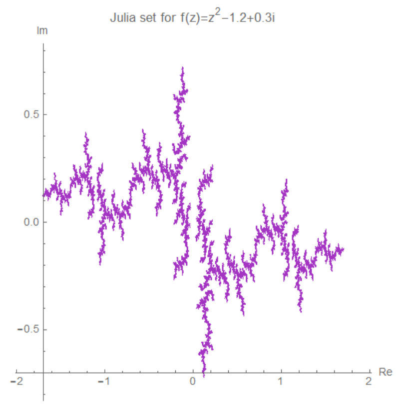
- We would now like to study the dynamics of a function $f : \mathbb{C} \to \mathbb{C}$ on the set of points where the orbits of f remain bounded, since ultimately this set is where all of the interesting behavior of f occurs.
  - We will primarily focus our attention on polynomial and rational maps, especially the quadratic family $q_c(z) = z^2 + c$, but the basic theory extends equally well to general holomorphic functions.

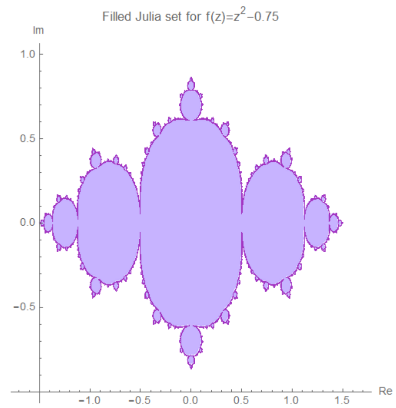
5.2.1 Definition of the Julia Set and Basic Examples

- Definition: If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial function, the filled Julia set of f is the set of points whose orbits are bounded. The Julia set of f, denoted J_f, is the boundary of the Julia set.
  - Recall that, if S is a subset of $\mathbb{C}$, the boundary of S, often denoted ∂S for shorthand, consists of all points $z \in \mathbb{C}$ such that any open ball of positive radius around z contains some points in S and some points not in S. (For sufficiently well-behaved regions the technical definition of "boundary" agrees with the colloquial use; namely, it is the set of points lying on the "edge" of the region.)
  - It turns out that for most functions, the Julia set will be rather complicated and have an irregular shape: indeed, it can even have an empty interior, so that the filled Julia set is the same as the Julia set itself.

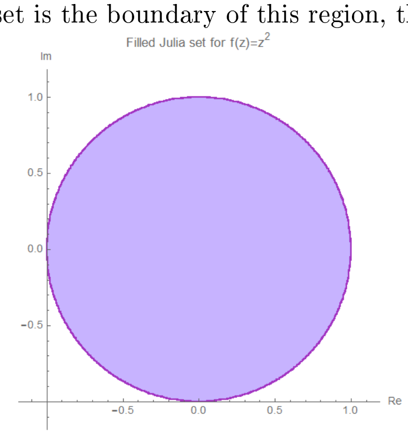
Here are some typical Julia sets:







- Example: Describe the orbits of the squaring map $q_0(z) = z^2$, as well as the Julia set and filled Julia set.
  - We have $q_0^n(z) = z^{2^n}$, so for $|z| < 1$ the orbit of z tends to the attracting fixed point $z = 0$, and for $|z| > 1$ the orbit of z tends to ∞.
  - It remains to analyze the orbits when $|z| = 1$: if $z = e^{i\theta}$, then $q_0(z) = e^{2i\theta}$, so, on the unit circle $|z| = 1$, the squaring map is the same as the angle-doubling map, whose behavior we already understand fairly well.
  - From our earlier analysis, we know that the angle-doubling map on the circle is chaotic, and its preperiodic points are the points $e^{i\theta}$ where θ is a rational multiple of 2π. In other words, the preperiodic points are the roots of unity; namely, the solutions to $z^k = 1$ for some $k \geq 1$.
  - The filled Julia set is the set of points whose orbit does not go to ∞, namely, the closed unit disc $\overline{D} = \{z : |z| \leq 1\}$, and the Julia set is the boundary of this region, the unit circle $\partial D = \{z : |z| = 1\}$:



  - By essentially the same analysis, we can in fact see that the nth power map $p(z) = z^n$ for any integer $n \geq 2$ will behave in essentially the same way: all orbits inside the open disc $|z| < 1$ tend to the attracting fixed point at $z = 0$, all orbits with $|z| > 1$ will tend to ∞.
  - On the unit circle, the nth power map is the same as the multiplication-by-n map on $\mathbb{R}$ modulo 1, and by the same arguments as above we see that the preperiodic points, the filled Julia set, and the Julia set for the nth power map are the same as for the squaring map.
- Example: Find the Julia set and filled Julia set for the map $q_{-2}(z) = z^2 - 2$.
  - We have already studied the behavior of this map on the real line: it is chaotic on the interval $[-2, 2]$, and outside this interval all orbits tend to $-\infty$.
  - It turns out that we have essentially found all of the interesting dynamical properties of this function already: we will show that the dynamical system (U_1, q_{-2}) is conjugate to (U_2, q_0), where U_1 is $\mathbb{C}$ with the real interval $[-2, 2]$ removed, and U_2 is the set $\{z : |z| > 1\}$.
  - In particular, this will imply that every orbit of q_{-2} except those lying in $[-2, 2]$ tends to ∞, since the same is true for the orbits of any point in U_2 under the squaring map q_0.

- We will show that $h(z) = z + \dfrac{1}{z}$ is a conjugacy from (U_2, q_0) to (U_1, q_{-2}):
  - For injectivity, suppose $w, z \in U_2$ and $h(z) = h(w)$: then $z + \dfrac{1}{z} = w + \dfrac{1}{w}$, which has the obvious solutions $w = z$, $w = 1/z$ and no others, because it is a quadratic equation in w (or just by factoring). If $z \in U_2$ then $1/z$ is not in U_2, so since $w \in U_2$ then we must have $w = z$, as required.
  - For surjectivity, if $\zeta \in U_1$ then we can find $z \in U_2$ with $h(z) = \zeta$ by solving the quadratic equation $z + \dfrac{1}{z} = \zeta$ to obtain $z = \dfrac{\zeta \pm \sqrt{\zeta^2 - 4}}{2}$. The product of these two solutions is 1, so either (i) one of them lies in U_2, or (ii) they both lie on the unit circle. In case (i) we are done, and in case (ii), if $z = e^{i\theta}$ then $\zeta = e^{i\theta} + e^{-i\theta} = 2\cos\theta$ is a real number in the interval $[-2, 2]$, which is excluded from U_1.
  - Clearly, $h(z)$ is continuous (since it is, in fact, holomorphic on U_2), and its inverse is also continuous by the chain rule (or equivalently, by the implicit function theorem, since h' is never zero on U_2).
  - Finally, we see that $h(q_0(z)) = h(z^2) = z^2 + z^{-2} = (z + z^{-1})^2 - 2 = q_{-2}(h(z))$, so h is a conjugacy as required.
- We will remark that this map does not simply spring out of nowhere: on the unit circle, we see that $h(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$, so this map is secretly the same as the (semi)-conjugacy we gave between the dynamical systems $([-2, 2], q_{-2})$ and the angle-doubling map.
- From all of this, we see that the filled Julia set (and thus also the Julia set itself) for f is simply the interval $[-2, 2]$ on the real line.

5.2.2 Computing the Filled Julia Set: Escape-Time Plotting

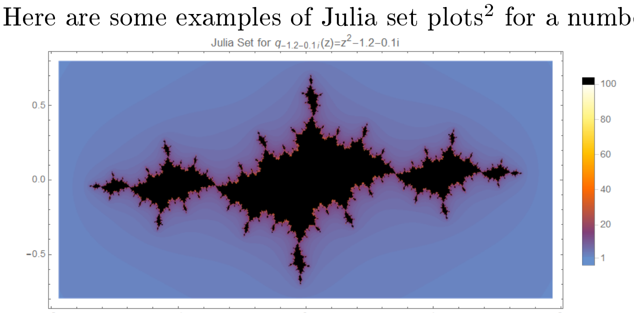
- In general it is very difficult to describe the Julia set of even simple functions like the quadratic maps $q_c(z) = z^2 + c$ except in very special cases like the two we analyzed earlier. We will now turn our focus toward describing computational methods for plotting Julia sets and filled Julia sets.
  - A natural method for plotting the filled Julia set is simply to test whether a given point in the plane has its orbit blow up to ∞ after some small number of iterations.
  - In order to make this more precise we will require an "escape criterion": namely, a criterion for when the orbit of a point under a map is guaranteed to blow up to ∞.
  - For polynomials, there is always an escape criterion that only depends on the absolute value of the starting point:
- Proposition (Polynomial Escape Criterion): Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is a polynomial of degree $n \geq 2$. Then there exists a constant $R > 0$, depending only on n and the coefficients a_i, such that the orbit of any point z with $|z| > R$ blows up to ∞.
  - Proof: We will show, more specifically, that for any $\lambda > 1$ there exists an R such that $|z| > R$ implies $|p(z)| \geq \lambda |z|$.
  - We can then repeatedly apply this result to see that $\left|p^d(z)\right| \geq \lambda^d |z| > \lambda^d R$, and the lower bound goes to ∞ by the assumptions $\lambda > 1$ and $R > 0$, so that $\left|p^d(z)\right| \to \infty$ as $d \to \infty$.
  - To show this, let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $n \geq 2$ and $a_n \neq 0$, and let $C = \displaystyle\sum_{i=0}^{n-1} |a_i|$.
  - We claim that if $|z| \geq \max(1,\, 2C/|a_n|,\, (2\lambda/|a_n|)^{1/(n-1)})$, then $|q(z)| \geq \lambda |z|$. Thus in particular, we can take $R = \max(1,\, 2C/|a_n|,\, (2\lambda/|a_n|)^{1/(n-1)})$.
  - So suppose $|z| \geq \max(1,\, 2C/|a_n|,\, (2\lambda/|a_n|)^{1/(n-1)})$. Then by repeatedly using the triangle inequality,

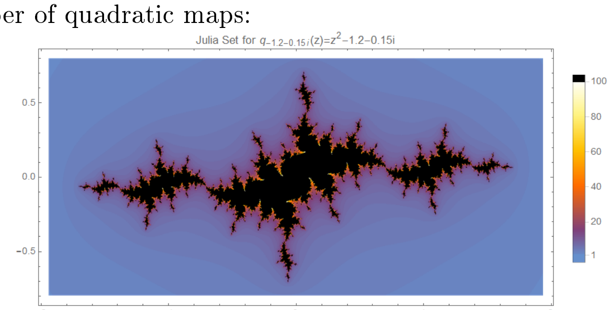
we have

$$\begin{aligned} |q(z)| &\geq |a_n z^n| - \left[\left|a_{n-1}x^{n-1}\right| + \cdots + |a_1 x| + |a_0|\right] \\ &\geq |a_n z^n| - \left[|a_{n-1}| + \cdots + |a_1| + |a_0|\right]\left|z^{n-1}\right| \\ &= |z^n| \cdot \left[|a_n| - C/|z|\right] \\ &\geq |z^n| \cdot \frac{1}{2}|a_n| = |z| \cdot \frac{1}{2}|a_n| \cdot |z|^{n-1} \\ &\geq |z| \cdot \lambda \end{aligned}$$

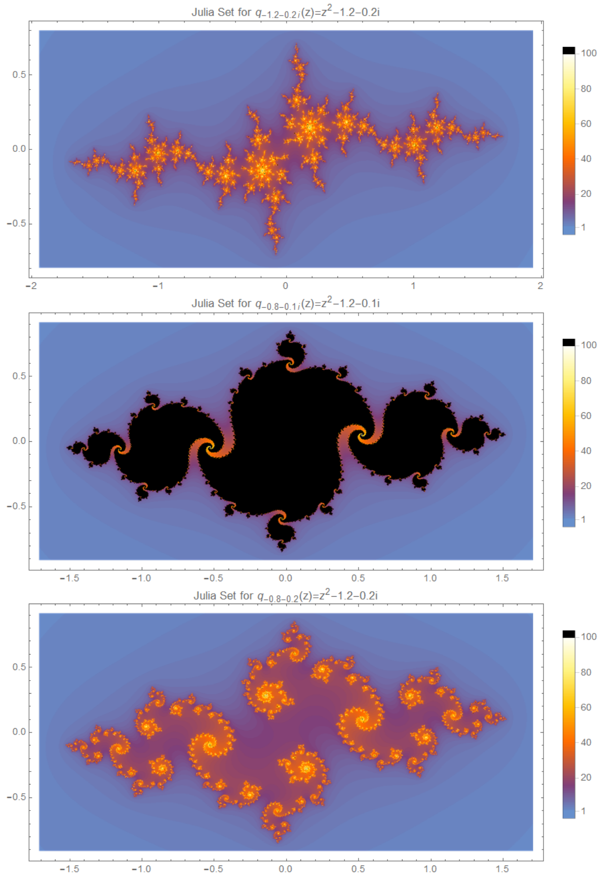
where the first inequality follows because $\left|z^{n-1}\right| \geq \left|z^k\right|$ for $k \leq n-1$ by the hypothesis that $|z| \geq 1$, the second inequality follows because $|z| \geq 2C/|a_n|$ so that $C/|z| \leq \frac{1}{2}|a_n|$, and the last inequality follows because $|z|^{n-1} \geq 2\lambda/|a_n|$.

- As an immediate corollary of this bound, we see that the Julia set for a polynomial map is always bounded since it is contained in the disc of radius R.
  - In general, the Julia set for more general types of maps, like rational functions, can be unbounded.
- For specific polynomial maps, we can sharpen the constant that appears in the proof:
- Proposition (Quadratic Escape Criterion): For the quadratic map $q_c(z) = z^2 + c$, if $|z| > \max(|c|, 2)$ then the orbit of z blows up to ∞.
  - The argument is essentially the same as the one we gave above for general polynomials, but sharpened slightly to take advantage of the low degree.
  - Proof: By assumption since $|z| > 2$ there is a $\lambda > 1$ such that $|z| - 1 > \lambda$.
  - Then by the triangle inequality, we have $|q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \geq \lambda |z|$.
  - Since $\lambda > 1$ this implies $|q_c(z)| \geq \lambda |z| > |z| > \max(|c|, 2)$.
  - Thus, $q_c(z) > \max(|c|, 2)$, so so we may iteratively apply the argument to see that $\left|q_c^d(z)\right| \geq \lambda^d |z|$, and since $\lambda > 1$ and $|z| > 2$ we conclude that $\left|q_c^d(z)\right| \to \infty$ as $d \to \infty$.
- Using the escape criterion we can give a method for plotting the filled Julia set:
- (Semi)-Algorithm (Escape Time): To plot the filled Julia set for the polynomial $p(z)$, let R be the escape radius given by the escape-criterion proposition. Choose a large number of points inside the disc $|z| < R$, evaluate the first 100 iterates of p on each point, and then color in each of the points that do not escape the disc $|z| < R$.
  - In principle, this procedure will completely characterize the points lying in the filled Julia set, but it is not always possible to check computationally whether a point will actually leave the disc $|z| < R$ within a finite amount of time, since there are points that will take an arbitrarily large number of iterations before leaving the disc.
  - We can also refine the picture slightly by plotting the points that escape the disc $|z| < R$ in various colors depending on how many iterations they require to leave the disc. In cases where the Julia set happens to have very little interior (or none at all) this procedure will help show the structure.
- Here are some examples of Julia set plots[2] for a number of quadratic maps:

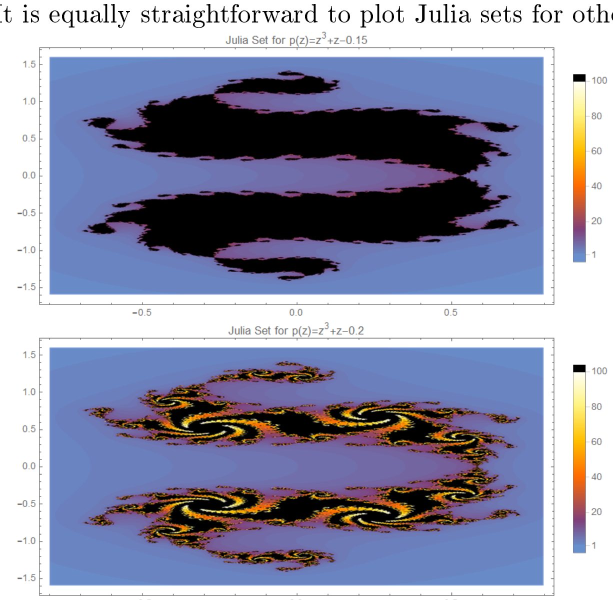




[2] Thanks to the coders of Mathematica 10 for implementing easy Julia set plotting.



- It is equally straightforward to plot Julia sets for other polynomial maps:



5.2.3 Computing the Julia Set: Backwards Iteration

- The escape-time plotting method allows us to plot the filled Julia set for a function $p(z)$, and in turn we can visually extract a rough picture of the Julia set itself as the boundary of the filled Julia set.
  - However, it would be nice to have an explicit way to plot the Julia set itself, without having to compute a large number of iterates of points most of which are not in the Julia set.

- Since the Julia set seems to have a fractal shape (much of the time, anyway), it is reasonably natural to hope that there might be a way to plot Julia sets using iterated function systems.
- Specifically, if we can exhibit the Julia set as the invariant set of a collection of contractions, or at least maps that are close to contractions, then it is reasonable to hope that we might be able to create an iterated function system related to the map p in some way.

• In the explicit examples we computed earlier, we saw that the set of repelling periodic points was dense in the Julia set. This turns out to be true in general:

• Theorem (The Julia Set as a Repelling Set): If f is any polynomial of degree ≥ 2, then every repelling periodic point of f is in the Julia set, and in fact every repelling preperiodic point of f lies in the Julia set as well.

- Proof (outline): It is easy to see that any periodic point (repelling or otherwise) certainly lies in the filled Julia set, since its orbit remains bounded. Indeed, all eventually periodic points will lie in the filled Julia set for the same reason.
- If z_0 is a repelling fixed point, then points in a sufficiently small disc around z_0 will have orbits that eventually land outside the disc: we claim in fact that some points in this disc must actually have orbits that blow up to ∞, meaning that z_0 is actually on the boundary of the filled Julia set (and thus, in the Julia set).
- For the last statement, it is sufficient to show the result when z_0 is a repelling fixed point. For simplicity, conjugate f by a translation to move z_0 to the origin, and then suppose $|f'(0)| = \lambda > 1$. By the chain rule, $|(f^n)'(0)| = |f'(0)|^n = \lambda^n$, so we have $f^n(z) = \lambda^n z + O(z^2)$ for sufficiently small $|z|$.
- Then it is possible to arrange matters so that $|f^n(z)| > \frac{\lambda^n}{2}|z|$ for some values of z, so by taking n large enough we can force $f^n(z)$ to land outside the escape radius of the function f, meaning that the orbit of z will escape to ∞. (We omit the rather delicate estimates required to show that this argument actually succeeds.)

• There is a natural extension of this result, due to Fatou and Julia:

• Theorem (Fatou-Julia): For any polynomial f of degree ≥ 2, the set of all repelling periodic points for f is a dense subset of the Julia set J_f.

- We will not prove this result, which relies on some deeper results of complex analysis involving normal families.
- However, it certainly agrees with the analysis we made for the function $f(z) = z^2$, whose Julia set is the unit circle $|z| = 1$: the periodic points were those points $e^{2\pi i(p/q)}$ where q was odd, and these points are dense on the unit circle.

• Corollary: If f is a polynomial, then the Julia set of f is completely invariant under f. In other words, $f(J_f) = J_f$ and $f^{-1}(J_f) = J_f$ also.

- Proof: Let S be the set of repelling periodic or eventually periodic points for f.
- If $z \in S$, then $f(z)$ is also a repelling eventually periodic point, so $f(z) \in S$.
- Similarly, any point in $f^{-1}(z)$ will be a repelling eventually periodic point, so $f^{-1}(z) \subset S$ as well.
- Since this holds for every $z \in S$ we see that $f(S) \subseteq S$ and $f^{-1}(S) \subseteq S$.
- Applying f to the second statement yields $S \subseteq f(S)$, so $f(S) \subseteq S \subseteq f(S)$. Equality must hold so $S = f(S)$, and thus $S = f(S) = f^{-1}(S)$.
- Now from the theorem of Fatou and Julia, if we take the topological closure of each of S, $f(S)$, and $f^{-1}(S)$, by the continuity of f and f^{-1} we obtain J_f, $f(J_f)$, and $f^{-1}(J_f)$ respectively. Since $S = f(S) = f^{-1}(S)$ we conclude $J_f = f(J_f) = f^{-1}(J_f)$.

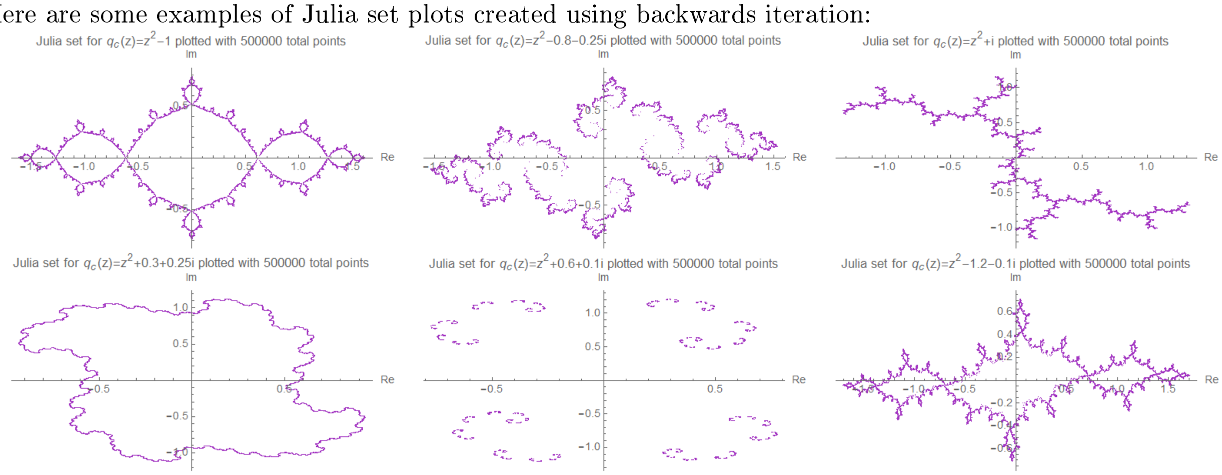
• This corollary suggests a way to compute the Julia set using iteration.

- Specifically, if f is a polynomial of degree d, then $f^{-1}(z)$, for most values of z, will be a set of d points, so we can think of $f^{-1}(z)$ as a collection of d different functions $\{g_1, g_2, \cdots, g_d\}$ of z.

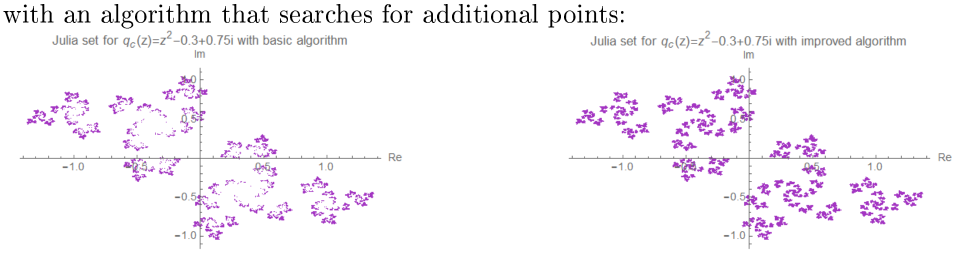
- Each of these component functions $g_i(z)$ will in general be a continuous function that is a contraction on a sufficiently large starting set: roughly speaking, since $|f(z)| \approx z^d$, we must have $|g_i(z)| \approx |z|^{1/d}$ for large $|z|$.
- By our results on iterated function systems, we immediately see that the Julia set is the invariant set of the iterated function system $\{g_1, g_2, \cdots, g_d\}$.
- There are some minor issues to work out regarding the "branch cuts" required to give a consistent definition to each function g_d, but this is the main idea.

- Algorithm (Backwards Iteration): To draw the Julia set for a polynomial $f(z)$, choose an arbitrary starting point $y_0 \in \mathbb{C}$, and then plot the points $\{y_0, y_1, y_2, y_3, y_4, \dots\}$ where y_m is obtained by randomly choosing one of the d points with $f(y_m) = y_{m-1}$. Equivalently, we take y_m to be a random one of the d backwards iterates of y_{m-1}.
  - It can be proven that this algorithm will converge to the Julia set for almost all starting points y_0.
    - The exceptions are called "exceptional points", and it can be proven that, for polynomial maps, they will only show up only for polynomials that are conjugate via a linear homeomorphism to a power map $z \mapsto z^n$ for some n.
    - It is easy to see that the Julia set for the power map is the circle $|z| = 1$, so exceptional points will only arise for polynomial maps whose Julia set is a circle. Thus, we do not lose anything by ignoring these cases, since we do not need the backwards iteration procedure in order to draw a circle!
  - In practice, one often chooses the starting point $y_0 = 0$. By the above remark, for every map in the quadratic family $q_c(z) = z^2 + c$ except for $q_0(z) = z^2$, the backwards iteration procedure applied to $y_0 = 0$ will indeed converge to the Julia set.
  - However, unlike with the iterated function systems of similarities we studied, the backwards iteration algorithm does not generally produce points that are evenly distributed through the Julia set: some parts of the Julia set are difficult to reach using the backwards iteration procedure, and so they will have comparatively few points plotted.

- Here are some examples of Julia set plots created using backwards iteration:



- It is also possible to refine the algorithm to sample more points near difficult-to-reach areas of the Julia set, to produce better pictures. Here is a comparison of two plots, one with the "naive" backward iteration method and another with an algorithm that searches for additional points:



- From all of the discussion above, we see that (J_f, f) is a dynamical system which has a dense set of periodic points, which is one of the three components in the definition of a chaotic dynamical system, and that all of the repelling periodic points lie in J_f, which is reminiscent of having sensitive dependence.
- Theorem (Chaotic Julia Sets): If f is a polynomial, then f is chaotic on its Julia set J_f.
  - This result also holds for other classes of functions, but the proof is more intricate.
  - Proof (outline): We already showed that f has a dense set of periodic points on J_f.
  - Transitivity follows from the fact (noted earlier) that for almost any point $y \in J$, the backward iterates of y are dense in J. By choosing appropriate elements in the chain of backward iterates, one can deduce that f is transitive.
  - Sensitive dependence follows because, for any point $z \in J$, there are repelling periodic points arbitrarily close to z: thus, $|f(z) - f(w)| > |z - w|$ for any w sufficiently close to z. Since J_f is compact by the assumption that f is a polynomial, there necessarily exists a constant $\lambda > 1$ and $r > 0$ such that $|f(z) - f(w)| > \lambda |z - w|$ for all $z, w \in J$ with $|z - w| < r$. Iterating this result shows that for any $z \neq w$, there is some k for which $\left|f^k(z) - f^k(w)\right| \geq r$: thus, f has sensitive dependence on J.

5.3 Additional Properties of Julia Sets, The Mandelbrot Set

- We will now briefly outline some deeper results about Julia sets for more general functions such as the rational functions $f(z) = p(z)/q(z)$ of degree $d \geq 2$ (the degree of a rational function is the maximum of the degrees of the numerator and denominator when written in lowest terms). Most of the results require significantly more theoretical background than we can develop at the moment, so we will focus primarily on giving a survey of important properties of Julia sets.
  - When possible we will attempt to motivate and explain each of the results using the examples we have already seen, as well as give some general discussion of the ingredients of the proofs.
- To cap our discussion, we will construct and study the Mandelbrot set, which is the set of points $c \in \mathbb{C}$ for which the map $q_c(z) = z^2 + c$ has a connected Julia set.

5.3.1 General Julia Sets

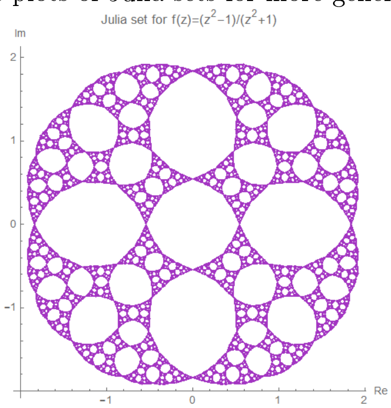
- The notion of the Julia set can be extended to general holomorphic functions, but the correct general definition is slightly different from the one we have given.
- One way is simply to define the Julia set as the closure of the set of repelling periodic points.
  - However, it is not so clear in general that most holomorphic functions have repelling periodic points at all (nor is it clear how to find them), so we will not take this as our definition.
- Definition: A family $\{f_n\}$ of holomorphic functions on an open set U is a normal family if every sequence of functions in the family has a subsequence which either converges uniformly on compact subsets of U, or converges uniformly to ∞ on U.
  - Roughly speaking, a normal family is a collection of maps which all either converge together or diverge to ∞ together, in a consistent way.
- Definition: If $f : U \to \mathbb{C}$ is a holomorphic function, the Fatou set of f is the set of points $z \in U$ on which the set of iterates $\{f^n\}$ is a normal family. The Julia set J_f of f is the set of points $z \in U$ on which the set of iterates $\{f^n\}$ fails to be a normal family.
  - We will remark that the definition above was actually the original historical definition of the Julia set.
  - Roughly speaking, the Fatou set is the set of points where f behaves in a predictable manner: the orbits of points near a point in the Fatou set stay bounded (in a predictable way) or blow up to ∞ (in a uniform way).

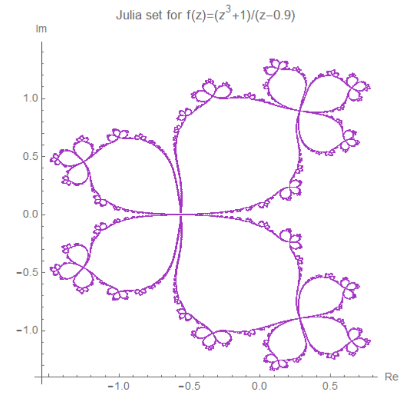
- The Julia set is the complement of the Fatou set.
- If f is a holomorphic function on the entire complex plane (e.g., if f is a polynomial or a function like e^z), then the Julia set J_f is the boundary of the set of points whose orbits escape to ∞.

• Algorithms for plotting Julia sets can be adapted, often with some difficulty, for many classes of general holomorphic maps.

- In general, most holomorphic functions will not have a simple escape criterion, and so instead we will have to resort to other procedures to generate escape-time plots. (To illustrate, notice that $|e^z| = e^{\mathrm{Re}(z)}$, so even when $|z|$is very large, e^z can be very close to zero.)
- The theorem on plotting Julia sets using backwards iteration will still mostly hold in the general setting, but again they can require some modification when implementing them computationally since Julia sets for general maps can extend to ∞.

• Here are some plots of Julia sets for more general maps:





• We will also remark that, although Julia sets often have a fractal shape and possess self-similarities, it is generally very difficult to compute their exact box-counting dimension.

5.3.2 Julia Sets and Cantor Sets

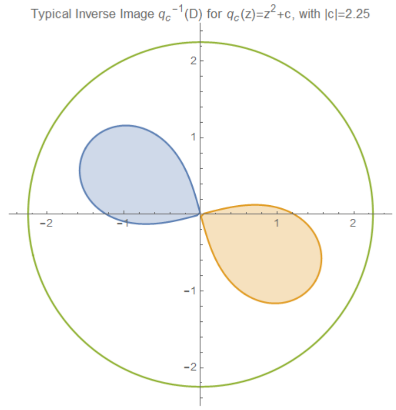
• When we studied the quadratic maps $q_c(x) = x^2 + c$ on the real line, we saw that when $c < -2$ the set of points where the orbit remained bounded was a Cantor set.

- In general, if S is a subset of a metric space, we say S is a Cantor set if S is homeomorphic to the Cantor ternary set.

• This is also true in the complex plane for values of c sufficiently far away from the origin:

• Theorem (Symbolic Dynamics in $\mathbb{C}$): If $|c|$ is sufficiently large, the Julia set J_c of the quadratic map $q_c(z) = z^2 + c$ is a Cantor set. More specifically, the dynamical system (J_c, q_c) is homeomorphic to (Σ_2, σ) where σ is the shift map on the binary sequence space Σ_2 whenever $|c| > \frac{5+2\sqrt{6}}{4} \approx 2.4747$.

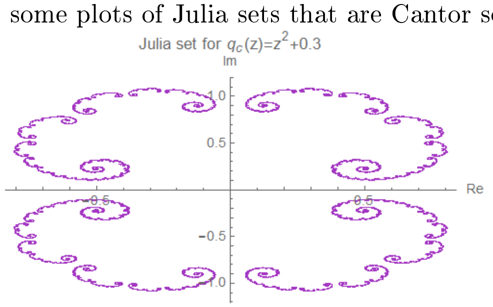
- The proof of this result is quite similar to the proof in the real case mentioned above: the only difference is that the nested sequences of intervals instead become nested sequences of sets resembling discs.
- We will also remark that one can prove a similar theorem for other families of polynomial maps of higher degree, although the statements and the results are correspondingly more complicated.
- Proof (outline): Assume $|c| > 2$. By the escape criterion, any point z with $|z| > |c|$ has orbit that blows up to ∞. Thus, any point outside the circle $|z| = |c|$ does not lie in the Julia set.
- Now let Λ be the set of points all of whose orbits lie inside the closed disc $D = \{z : |z| \le |c|\}$. Then Λ is given by the infinite intersection $\Lambda = \bigcap_{n=1}^{\infty} q_c^{-n}(D)$, so it suffices to study these inverse image regions.
- One can compute the preimage $q_c^{-1}(D)$ to see that it is a "figure-eight" region lying inside D that consists of two lobes joined at the origin, one copy for each of the two possible choices of square root:

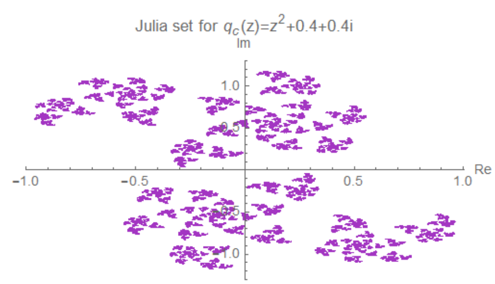


- It is then mostly straightforward to verify that $q_c^{-n}(D)$ is the union of 2^n distinct regions in the plane (which we can label using the appropriate binary string of length n in the same manner as we did with the intervals in the real case).
- Now define an itinerary map for a point lying in the region Λ using the two lobes D_0 and D_1 of the figure-eight: if $z \in \Lambda$, set $S(z) = (d_0 d_1 d_2 \cdots)$, where $d_n = 0$ if $q_c^n(x) \in D_0$ and $d_n = 1$ if $q_c^n(x) \in D_1$.
- Using a higher-dimensional analogue of Cantor's nested intervals theorem and the labelings we have attached to the components of $q_c^{-n}(D)$, we can show that the itinerary map is a bijection, and we can also verify that the map is continuous and has continuous inverse using the definition of continuity provided that $|c|$ is sufficiently large.
- Furthermore, if $|c|$ is large enough, then it will be the case that $|q_c'(z)| > 1$ on each of the regions: if B is the disc of radius $1/2$ centered at the origin, then $q_c(B)$ is the disc of radius $1/4$ centered at c. For large enough c, this disc will be disjoint from $q_c^{-1}(D)$, since the lobed region only extends a distance $\sqrt{2\,|c|}$ away from the origin. This disc contains all the points such that $\left|q_c'(q_c^{-1}(z))\right| \le 1$, so any point in Λ not lying in this disc will have $|q_c'(z)| > 1$.
- So since $|q_c'(z)| > 1$ on all of Λ, all of the periodic points in Λ will be repelling. Since the periodic points in Σ_2 are dense, we conclude that the periodic points in Λ are also dense, and so Λ must actually be the Julia set for q_c.
- Finally, for the explicit bound on c, we can in fact see that it is sufficient to assume that $|c| - \frac{1}{4} > \sqrt{2\,|c|}$, which (it is straightforward to check) is equivalent to $|c| > \frac{5+2\sqrt{6}}{4}$.

- As a corollary of the result above, we see that if $|c|$ is sufficiently large, then the Julia set J_c for $q_c(z) = z^2 + c$ will be homeomorphic to the Cantor ternary set.
  - Thus, the Julia set will be totally disconnected, so in particular this means that the filled Julia set and the Julia set are necessarily the same.
  - Furthermore, if we try to plot the filled Julia set using the escape-time method, we will essentially never be able to identify any points that actually lie in the set itself due to rounding errors.
  - This is why, when drawing the filled Julia set, we do not simply plot the points whose orbits remain bounded, but also points whose orbits escape based on how long their escape takes: when the Julia set is a Cantor set, we are unlikely to find any points at all whose orbits do not eventually escape, so we would not plot anything in our picture.

- Here are some plots of Julia sets that are Cantor sets:



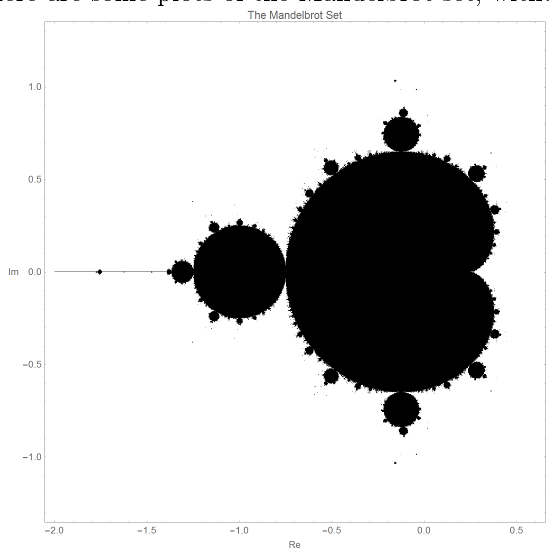


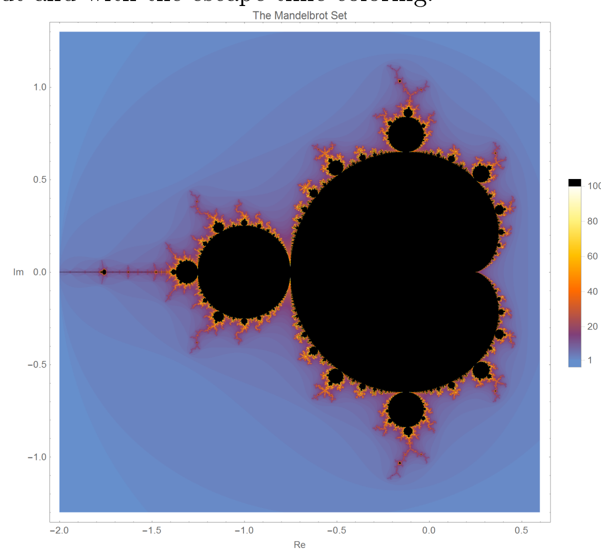
5.3.3 Critical Orbits and the "Fundamental Dichotomy"

- For real-valued functions with negative Schwarzian derivative, we know that every attracting cycle of f attracts at least one critical point of f. There is a corresponding (and simpler) result for holomorphic functions:
- Theorem (Critical Orbits): If $\{z_1, \cdots, z_n\}$ is an attracting cycle of the rational function f of degree $d \geq 2$, then its basin of attraction is either infinite or contains at least one critical point of f. If f is a polynomial of degree d, then it has at most $d-1$ attracting cycles.
  - We remark that the count of critical points may include ∞ as a critical point, depending on the relative degrees of the numerator and denominator of f. (We define $f(\infty) = \lim_{|z|\to\infty} f(z)$ if this limit exists.)
  - Like in the real case, it is sufficient to prove the result for the attracting basin of an attracting fixed point of f, since the chain rule allows us to convert a statement about the attracting basin for a fixed point of f^n to a statement about the attracting basin for an n-cycle of f.
  - The proof of this theorem is quite different from the real case, and again requires some rather deep results from complex analysis. (Note that we no longer require any information about the Schwarzian derivative of the function f, for example.) We will omit the details.
- As a corollary of the previous theorem, we see that $q_c(z) = z^2 + c$ has at most one attracting cycle, and that if there is an attracting cycle it will attract the orbit of the critical point $z = 0$.
- In fact, whether $q_c(z)$ possesses an attracting cycle is closely related to whether the Julia set is a Cantor set:
- Theorem (The Fundamental Dichotomy): For $q_c(z) = z^2 + c$, either (i) the orbit of 0 escapes to ∞ in which case the Julia set of q_c is a totally disconnected Cantor set consisting of infinitely many disjoint components, or (ii) the orbit of 0 remains bounded in which case the Julia set of q_c has a single connected component.
  - A set S is connected if it cannot be written as the disjoint union of two (relatively) closed subsets: in other words, if there do not exist closed subsets C_1 and C_2 of $\mathbb{C}$ such that $(S \cap C_1)$ and $(S \cap C_2)$ have union S and empty intersection. (Roughly speaking, a connected set cannot be split apart into two separate pieces that do not touch one another.)
  - More generally, if p is a polynomial then the Julia set J_p is totally disconnected if every critical orbit escapes to ∞, and J_p is connected if every critical orbit is bounded.
  - Proof (outline): Let D_r be a disc of arbitrary radius $r > 2$. By the escape criterion, all points on the boundary of D_r escape to ∞.
  - It is a straightforward geometric calculation that if R is a connected region in $\mathbb{C}$, then $q_c^{-1}(R)$ is connected if R contains c and $q_c^{-1}(R)$ consists of two connected pieces if R does not contain c.
  - Now consider the iterated inverse images $q_c^{-1}(D_r)$, $q_c^{-2}(D_r)$, ...: the filled Julia set is necessarily the intersection of these sets, since any point lying in the intersection will never escape, and conversely any point that lies outside one of these sets will escape.
  - Suppose first that the orbit of 0 escapes to ∞. Then $\left|q_c^k(0)\right| > 2$ for some k. Set $r = \left|q_c^k(0)\right|$: then each of $q_c^{-1}(D_r)$, $q_c^{-2}(D_r)$, ... , $q_c^{-k}(D_r)$ is a single region bounded by a simple closed curve, but $q_c^{-(k+1)}(D_r)$ will consist of two separate regions (since it will not contain the point c).
  - Each additional inverse iterate will then double the number of connected components, and it can also be shown that the size of each component goes to zero as the number of inverse iterates grows.
  - So by the 2-dimensional version of Cantor's nested intervals theorem, the filled Julia set consists of an infinite number of disjoint components and is totally disconnected. Since it is totally disconnected, every point lies on the boundary, so the filled Julia set and the Julia set are the same.
  - Finally, we can define an itinerary map in essentially the same way as we did before to show that the filled Julia set is homeomorphic to the Cantor ternary set.
  - Now suppose that the orbit of 0 does not escape to ∞. Let $r > 2$. Since the orbit of 0 does not escape, $q_c^k(0)$ never lies outside the circle of radius 2. Thus, $q_c^{-k}(D_r)$ always contains $q_c(0) = c$ for every value of k.
  - So, for each k, we see that $q_c^{-k}(D_r)$ consists of a single connected component.
  - It then follows from the 2-dimensional version of Cantor's nested intervals theorem that the intersection of a nested sequence of connected closed sets is connected, so we immediately see that the filled Julia set is connected. It can be shown that this implies the Julia set itself is also therefore connected.

5.3.4 The Mandelbrot Set

- From the fundamental dichotomy, we see that the Julia set for the map $q_c(z) = z^2 + c$ is either a totally disconnected Cantor set or a set with a single connected component, according to whether the orbit of 0 escapes to ∞ or not.
- We can graphically depict the different cases by plotting the values of c for each possibility:
- Definition: The Mandelbrot set is the set of values of $c \in \mathbb{C}$ such that the orbit of 0 under $q_c(z) = z^2 + c$ does not escape to ∞.
  - The Mandelbrot set is often displayed using the same escape-time algorithm used for plotting filled Julia sets: namely, choose a large number of values of c inside the disc $|c| < 2$, evaluate the first 100 iterates of 0 under q_c for each c, and then color in the values of c for which the orbit of 0 does not escape the disc of radius 2. (We can further refine the picture by plotting points that do escape in a color indicating how many iterations were required.)
  - Here are some plots of the Mandelbrot set, without and with the escape-time coloring:





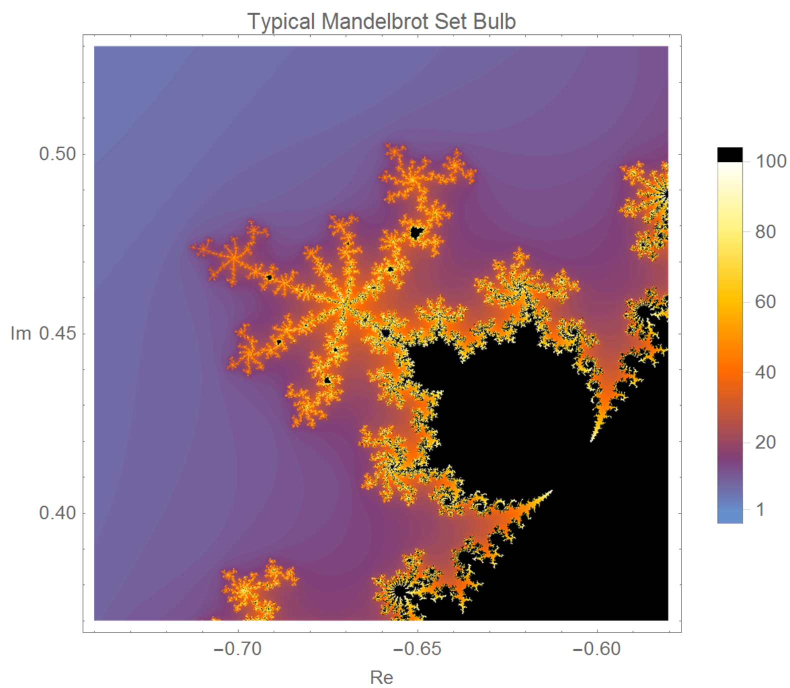
- Using our previous results we can establish a few basic properties of the Mandelbrot set:
  - By the escape criterion, we can equivalently characterize the Mandelbrot set as the set of points $c \in \mathbb{C}$ for which $|q_c^n(0)| \leq 2$ for all $n \geq 0$. In particular, since the set $|q_c^n(0)| \leq 2$ is closed for each n, the Mandelbrot set is an intersection of closed sets and is therefore closed.
  - Since $q_c(0) = c$, if $|c| > 2$ then the orbit of 0 will necessarily escape to ∞ by the escape criterion. Thus, the Mandelbrot set is contained inside the circle of radius 2 centered at 0.
  - From the fundamental dichotomy, the Mandelbrot set is also the set of values of c for which the Julia set J_c is connected.
  - The intersection of the Mandelbrot set with the real axis is the interval $[-2, 1/4]$: this result follows immediately from our previous analysis of quadratic maps on the real line. Explicitly, if $c > 1/4$ then all real points have orbit diverging to ∞, and if $c < -2$ then 0 escapes by the escape criterion. For $-2 \leq c \leq 1/4$ we know that the orbit of 0 will remain bounded, since q_c maps the interval between the two fixed points into itself, and 0 lies in that interval.
- There are a number of very deep theorems about the structure of the Mandelbrot set (and about the closely related Julia sets for quadratic maps). Here are a few:
- Theorem (Douady/Hubbard, 1982): The Mandelbrot set is connected.
  - It is somewhat plausible based on the picture to conjecture that the Mandelbrot set is connected, as the escape-time algorithm suggests the existence of "tendrils" that join the small bulbs to the main cardioid.

- In fact, Mandelbrot originally conjectured that the Mandelbrot set was disconnected – although it should be noted that he was only working with lower-resolution monochrome pictures that did not make use of the escape-time method, and he revised his original conjecture once better algorithms were available.
- Although the Mandelbrot set is known to be connected, it is still an open question whether it is locally connected (i.e., if for every open subset V of the Mandelbrot set and any $x \in V$, there exists a connected open set U with $x \in U$ that is contained in V).

- Theorem (Shishikura, 1998): The Hausdorff dimension (and therefore also the box-counting dimension) of the boundary of the Mandelbrot set is 2.
  - Although the boundary of the Mandelbrot set has a fractal appearance, much like Julia sets, it is actually sufficiently complicated that it has dimension 2, the same dimension as the plane itself.

5.3.5 Attracting Orbits and Bulbs of the Mandelbrot Set

- From the theorem about critical orbits, if q_c has an attracting cycle, then 0 will be attracted to it, and therefore cannot go to ∞. Thus, if q_c has an attracting cycle, then c lies in the Mandelbrot set.
  - In particular, we can see that the Mandelbrot set contains all points inside the curve $z = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$, since these values of c produce an attracting fixed point for q_c. This part of the Mandelbrot set is called the main cardioid.
  - We also see that the Mandelbrot set contains the points inside the circle of radius $1/4$ centered at -1, since these values of c produce an attracting 2-cycle for q_c.
  - In a similar way, there are further bulbs (also called "decorations") attached to the main cardioid, and to other bulbs, containing values of c producing attracting cycles of larger periods.
  - Each bulb consists of a main disc that in turn has additional smaller "decorations" attached to it, including a large antenna that branches off in roughly the opposite direction to the main cardioid.
  - Here is a typical Mandelbrot bulb:



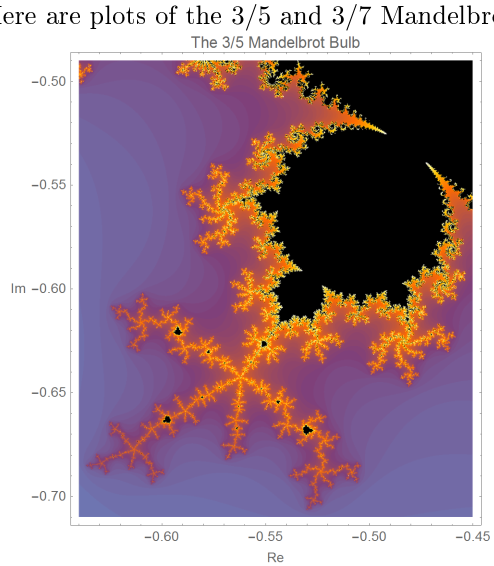
- Theorem (Mandelbrot Bulb Labeling): For each rational number p/q in lowest terms strictly between 0 and 1, there is a bulb attached to the main cardioid of the Mandelbrot set at the point $\frac{1}{2}e^{2\pi i(p/q)} - \frac{1}{4}e^{4\pi i(p/q)}$. For any c inside the main disc of this bulb, the function q_c has an attracting cycle of exact period q, and this cycle has rotation number p/q. (We call this bulb the p/q-bulb.) The size of the p/q bulb depends primarily on q, and shrinks as q grows.

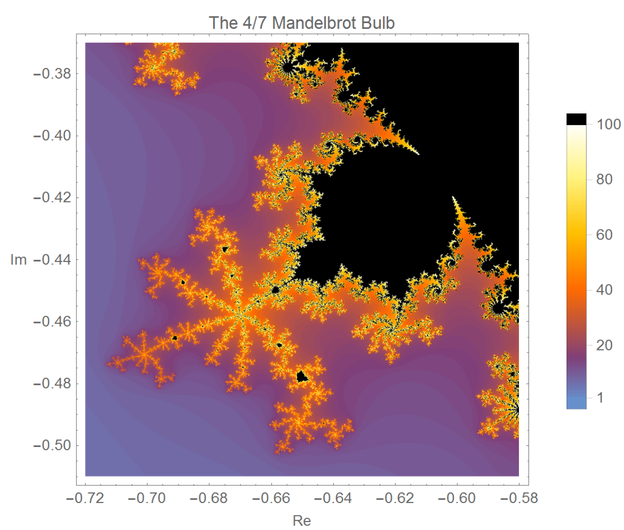
- We say a periodic cycle for a function f has rotation number p/q if each iteration of f rotates by p/q full rotations on average. (The prototypical example is the fixed point at 0 of the function $f(z) = e^{2\pi i(p/q)} z$.)
- We will not prove any of the parts of this theorem. However, for $z_0 = \frac{1}{2}e^{2\pi i(p/q)} - \frac{1}{4}e^{4\pi i(p/q)}$, we can see that $q'_{c_0}(z_0) = e^{2\pi i(p/q)}$, meaning that z_0 is a neutral fixed point with rotation number p/q.
- Roughly speaking, as we move out from c_0 into the main disc of the bulb, the neutral fixed point z_0 splits into an attracting q-cycle each point of which has rotation number p/q.

- Corollary: If a/b and c/d are consecutive bulbs on the main cardioid of the Mandelbrot set, then the largest bulb that appears between them will be the $(a+c)/(b+d)$ bulb.
  - There are a number of connections between the bulb labelings and the so-called Farey sequences of rational numbers. The nth Farey sequence is obtained by writing all rational numbers in $[0,1]$ whose denominators are $\leq n$ (in lowest terms) in increasing order.
  - Thus, for example, the 4th Farey sequence is $\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$.
  - From the bulb-labeling theorem, if we read the nth Farey sequence in order, we will obtain an ordered list of the largest bulbs around the perimeter of the main cardioid (where we interpret $\frac{0}{1}$ and $\frac{1}{1}$ as referring to the main cardioid itself).
  - In particular, since the largest bulbs will correspond to terms with the smallest denominators, the corollary follows from the elementary number theory fact that if a/b and c/d are consecutive members of the nth Farey sequence, then the first term in any subsequent Farey sequence that appears between them is $(a+c)/(b+d)$.

- It also turns out that we can identify the rotation number p/q of the bulb (and the period) using the geometric shape of the antenna:
- Theorem (Mandelbrot Bulb Antenna): The antenna of the p/q bulb has precisely q spokes emanating from its central point (including the spoke joining the bulb to the antenna). Furthermore, if the spokes are labeled $0, 1, \ldots, q-1$ counterclockwise starting with a 0 on the spoke joining the bulb to the antenna, the smallest spoke has label p.
  - Again, we will not prove this theorem[3], but instead settle for giving some suggestive examples.
  - We remark that the correct definition of "smallest spoke" does not always quite agree with the Euclidean distance in the plane (and it is difficult to define "smallest" in a precise enough way to prove the theorem). However, in practice, the spoke that appears visually smallest is usually the correct one.

- Here are plots of the $3/5$ and $3/7$ Mandelbrot bulbs as a demonstration of the antenna theorem:

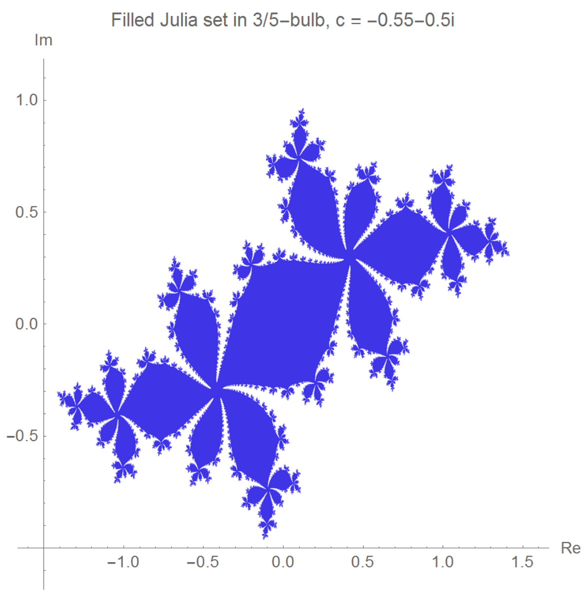


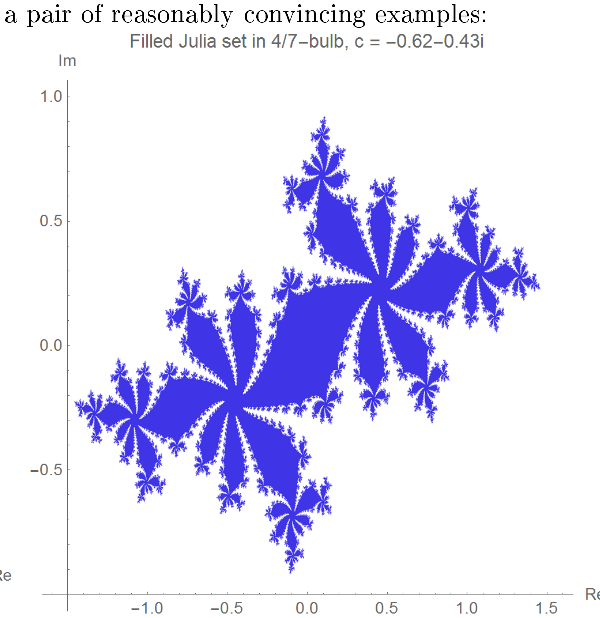


[3]The paper "The Mandelbrot Set, the Farey Tree, and the Fibonacci Sequence" by R. Devaney, from the April 1999 issue of the American Mathematical Monthly, has an accessible discussion of the proof of each of the results we have discussed.

- Observe that for the $3/5$ bulb, the main antenna has five spokes, and the third one (counterclockwise from the one connecting the antenna to the main disc) is the smallest. Also, by the bulb-labeling theorem, the bulb should be attached to the main cardioid at the point $c_0 = \frac{1}{2}e^{2\pi i(3/5)} - \frac{1}{4}e^{4\pi i(3/5)} \approx -0.482 - 0.532i$, which indeed it appears to be.
- Similarly, for the $4/7$ bulb, the main antenna has seven spokes, and the fourth one (counterclockwise from the one connecting the antenna to the main disc) is the smallest. The bulb should be attached to the main cardioid at $c_0 = \frac{1}{2}e^{2\pi i(4/7)} - \frac{1}{4}e^{4\pi i(4/7)} \approx -0.606 - 0.412i$, which indeed it appears to be.

- To close our discussion, we will mention one final connection between the p/q-labeling of the Mandelbrot bulb and the structure of the filled Julia set for points lying in the main disc of the bulb:
- Theorem (Filled Julia Set for Main Disc Points): If c is any point lying in the main disc of the p/q bulb of the Mandelbrot set, then the filled Julia set for q_c possesses infinitely many "junctions" at which exactly q lobes of the filled Julia set narrow and join together. If the lobes are labeled $0, 1, \ldots, q-1$ clockwise starting with a 0 on the largest lobe, the second-largest lobe has label p.
  - We will not prove this theorem either, but here is a pair of reasonably convincing examples:





  - In the first filled Julia set (for a c-value in the $3/5$ bulb), observe that there are 5 "lobes" centered around each point, and if we label the largest one 0 and proceed clockwise, then the second-largest has label 3.
  - Similarly, in the second filled Julia set (for a c-value in the $4/7$ bulb), observe that there are 7 "lobes", and if we label the largest one 0 and proceed clockwise, then the second-largest is indeed labeled 4.

Well, you're at the end of my handout. Hope it was helpful.