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1. Each item was worth 2.5 points.

   (a) We can see that there is a single segment in the diagram, indicating that f_λ has an attracting fixed point for $0.5 < \lambda < 1$.

   (b) At $\lambda = 1$ the single curve splits into two curves, indicating that there is a period-doubling bifurcation at $\lambda = 1$: the attracting fixed point becomes repelling and creates an attracting 2-cycle in its place.

   (c) We can see there are two points in the diagram, indicating that f_λ has an attracting 2-cycle on the range $1 < \lambda < 1.5$.

   (d) At $\lambda = 1.5$ each of the two curves splits into two, indicating that the 2-cycle has a period-doubling bifurcation: the attracting 2-cycle becomes repelling and creates an attracting 4-cycle in its place.

   (e) From the diagram it appears that there is an attracting 3-cycle on the range $2 < \lambda < 2.01$.

   (f) Yes, for $1.99 < \lambda < 2$ it looks like f has chaotic behavior, since the picture consists of a continuous range of points. For some λ in the range $2.03 < \lambda < 2.04$ the behavior also seems chaotic, but on a union of several intervals.

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2. Each part was worth 4 points.

   (a) Notice that **a** and **b** differ in the 0th, 3rd, 5th, 7th, ... places, so computing the distance sum yields $d(\mathbf{a}, \mathbf{b}) = \frac{1}{2^0} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots = \boxed{1 + \frac{1/2^3}{1 - 1/2^2} = \frac{7}{6}}$.

   (b) The points of period n are the repeating n-digit binary strings $(\overline{d_1 d_2 d_3 \cdots d_n})$, where the string does not repeat with any period smaller than n. Some examples of period-6 points are $(\overline{000001})$ and $(\overline{101001})$, and some examples of period-7 points are $(\overline{0000001})$ and $(\overline{0111010})$.

   (c) There are $2^7 - 2 = 126$ points of exact period 7 (all strings of length 7, minus the all-0 and all-1 strings since those have period 1. These group into cycles of length 7, so the number of cycles is $(2^7 - 2)/7 = \boxed{18}$. Alternatively, we could use the Mobius formula discussed in class.

   (d) The orbit is $\boxed{\text{not dense}}$, because there are many points in Σ_2 that cannot be the sequence of elements in the orbit. For example, no element in the orbit of **c** will have two consecutive zeroes, so there is no element in the orbit of **c** that is within a distance $1/4$ of the point $(00000 \cdots)$.

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3. Part (a) was worth 6 points while part (b) was worth 4 points.

   (a) We have $g(h(x)) = g(ax+b) = (ax+b)^2 - 2 = a^2x^2 + 2abx + (b^2 - 2)$ while $h(f(x)) = a(4x^2 - 4x + 1) + b = 4ax^2 - 4ax + (a + b)$. Comparing coefficients yields $a^2 = 4a$, $2ab = -4a$, and $b^2 - 2 = a + b$. Since we need $a \neq 0$ the first equation gives $a = 4$ while the second equation gives $b = -2$, and this works in the third equation. So we take $h(x) = \boxed{4x - 2}$.

   (b) We can see that the linear function h from part (a) is a homeomorphism from $[0, 1]$ to $[-2, 2]$ since is is continuous, a bijection, and has continuous inverse $h^{-1}(x) = (x + 2)/4$. Since our spaces are infinite and $g(x)$ is chaotic on $[-2, 2]$, by our theorem from class, that means f is chaotic on $[0, 1]$.

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4. Each item was worth 1.5 points.

   (a) $\boxed{\text{False}}$: This set is not dense since for example the value 1 cannot be written as limit of a sequence of points in the set.

   (b) $\boxed{\text{False}}$: The point $x = 0$ is an attracting fixed point of f, so f does not have sensitive dependence there (nearby orbits do not move away from each other).

   (c) $\boxed{\text{True}}$: The point $x = 1$ is a repelling fixed point of f, so f does have sensitive dependence there (nearby orbits are pushed away from $x = 1$).

   (d) $\boxed{\text{False}}$: This orbit is not transitive; since the point is periodic, there are only three distinct values in its orbit, so it does not approach arbitrary points in the sequence space.

   (e) $\boxed{\text{False}}$: Although a chaotic function must have a dense set of periodic points, there is no requirement that it have a point of period 3 (in fact, there exists a tent map that has no point of period 3 but is still chaotic).

   (f) $\boxed{\text{True}}$: by Sarkovskii's theorem, since any chaotic function has points of many different periods and 1 is at the end of the Sarkovskii ordering, the chaotic function must have a fixed point.

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5. Each item was worth 5 points.

   (a) Note that g is continuous since the values of the two pieces agree when $x = 1$. Additionally since $g(0) = 1$, $g(1) = 2$, and $g(2) = 0$, g has a 3-cycle. Hence by the period-3 theorem (or Sarkovskii's theorem), g has a cycle of period n for all n.

   (b) By Sarkovskii's theorem, h must also have a point of all periods following 10 in the Sarkovskii ordering, but not necessarily any values preceding 10. These values are the even integers other than 6, along with 1. The excluded integers are 6 and odd numbers greater than 1.

   (c) In fact, every point on the unit circle is a period-3 point for h, because iterating h three times yields the identity function, but h does not fix any points on the circle. So h certainly has points of exact period 3 but no other periods. This does not contradict Sarkovskii's theorem because Sarkovskii's theorem only applies to functions defined on an interval of the real line.

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6. Part (a) was worth 7 points while (b) and (c) were each worth 4 points.

   (a) Observe that each stage of the construction turns each square into 4 smaller squares of 1/3 the side length. Thus, in the nth stage of the construction there are 4^n squares each of side length $1/3^n$. The area of each square is $(1/3^n)^2 = 1/9^n$ so the total area is $\boxed{4^n/9^n}$, and the perimeter of each square is $4/3^n$ so the total perimeter is $\boxed{4 \cdot 4^n/3^n}$. As $n \to \infty$ the area tends to 0 while the perimeter tends to ∞.

   (b) The topological dimension is 0. To show this, observe that for any square in any stage of the construction, we can take a circle with the same center that is slightly larger than the circumcircle of that square, then the circle will not intersect the set. Thus, for any point in the Cantor dust fractal, we can find a small neighborhood of that point (namely, the disc representing the interior of the circle described above) whose boundary (namely, the circle) does not intersect the set. This satisfies the requirement for showing the topological dimension is 0.

   (c) If we just consider the x-coordinates of points in the Cantor dust fractal, we see that each stage removes the open middle third of each remaining interval, and the same holds for y-coordinates. Therefore, the nth stage of the Cantor dust fractal consists of points (x, y) whose coordinates are both in the nth stage of the Cantor set. Taking the intersection of all stages yields the result immediately.
