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1. Parts (a) and (b) were 4 points while (c) was 6 points.

   (a) We have $h(1) = 2$, $h(2) = -1$, $h(-1) = 0$, $h(0) = 1$, so it lies in the 4-cycle $\boxed{\{1, 2, -1, 0\}}$.

   (b) From the table we have $(h^4)'(0) = h'(1) \cdot h'(2) \cdot h'(-1) \cdot h'(0) = 0$, so the cycle is $\boxed{\text{attracting}}$.

   (c) We use the intermediate value theorem on $g(x) = h(x) - x$, since it is continuous.

   - We have $h(-2) = -15$ and $h(0) = 1$, so h has a fixed point in $(-2, 0)$ by the intermediate value theorem.
   - Likewise $h(1) = 1$ and $h(2) = -3$, so h has a fixed point in $(1, 2)$ by the intermediate value theorem.

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2. Part (a) was worth 4 points while (b) and (c) were worth 5 points each.

   (a) We have $\frac{2}{9} \to \frac{4}{9} \to \frac{8}{9} \to \frac{7}{9} \to \frac{5}{9} \to \frac{1}{9} \to \frac{2}{9}$ so the period is $\boxed{6}$. Also, $\frac{1}{15} \to \frac{2}{15} \to \frac{4}{15} \to \frac{8}{15} \to \frac{1}{15}$ so the period is $\boxed{4}$.

   (b) Since $D(x) = 2x$ modulo 1, $D^n(x) = 2^n x$ modulo 1. Therefore, $D^n(\frac{k}{2^n}) = 0$, so after n iterations of the function we will reach the fixed point 0.

   (c) The orbit of $\frac{1}{2^n - 1}$ is $\frac{1}{2^n - 1} \to \frac{2}{2^n - 1} \to \frac{4}{2^n - 1} \to \cdots \to \frac{2^{n-1}}{2^n - 1} \to \frac{1}{2^n - 1}$. So it is periodic with period n as claimed. This is true for every $n \geq 2$ so D has an n cycle for those n, and it also has the 1-cycle $\{0\}$.

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3. Part (a) was worth 8 points while (b) was worth 6 points.

   (a) First we find the fixed points, then we classify them.

   - Using the table we can see two fixed points $x = 0$ and $x = -1$.
   - To find all fixed points we must solve $x = \frac{3}{2}x - \frac{1}{2}x^2 - x^3$ or equivalently $\frac{1}{2}x - \frac{1}{2}x^2 - x^3 = 0$. We can then factor using the two known solutions $x = 0$ and $x = -1$ to obtain $\frac{1}{2}x(1 + x)(1 - 2x) = 0$ so that $x = \boxed{-1, 1/2, 0}$ are the fixed points.
   - Since $f'(x) = 3/2 - x - 3x^2$, we see $f'(0) = 3/2$, $f'(-1) = -1/2$, and $f'(1/2) = 1/4$.
   - Thus, $\boxed{0 \text{ is repelling}}$ while $\boxed{-1 \text{ and } 1/2 \text{ are attracting}}$.

   (b) We will find the set of possible endpoints: either the endpoints are the given 2-cycle, or one of them is a fixed point.

   - Since -1 and $1/2$ are both attracting, neither can be an endpoint of the other's immediate basin as noted in a homework problem (otherwise their basins would overlap, which is impossible). Thus the only possible finite endpoints are 0 or one of its preimages.
   - To find the preimages of 0, we solve $\frac{3}{2}x - \frac{1}{2}x^2 - x^3 = 0$, which using the table we see has two obvious solutions $x = 0$ and $x = 1$. Using them to factor we obtain $\frac{1}{2}x(3 + 2x)(1 - x) = 0$ so the other two preimages are $x = -3/2$ and $x = 1$.
   - Thus our possible endpoints are $-\infty$, -1.7867, $-3/2$, 0, 1, 1.4275, $+\infty$.
   - Since -1 lies between $-\frac{3}{2}$ and 0 the immediate basin of 0 is $\boxed{(-3/2, 0)}$, and since $1/2$ lies between 0 and 1, the immediate basin of $1/2$ is $\boxed{(0, 1)}$.

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4. Parts (a) and (b) were worth 3 points each, while (c) was worth 4 points.

   (a) By definition, $N(x) = x - \dfrac{f(x)}{f'(x)} = \boxed{x - \dfrac{x^4 - x^3}{4x^3 - 3x^2}}$.

   (b) The zeroes of f are $x = \boxed{0,\ 1}$. From the factorization of f we see $x = 0$ has $\boxed{\text{multiplicity } 3}$ and $x = 1$ has $\boxed{\text{multiplicity } 1}$.

   (c) From Newton's fixed point theorem, we know that a root of multiplicity 1 will have superexponential convergence, and a root of multiplicity higher than 1 will only have exponential convergence. Since $x = 0$ has multiplicity 3, it has exponential convergence, while $x = 1$ has multiplicity 1 and thus has superexponential convergence. So the convergence to $\boxed{x = 1}$ will be faster.

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5. Each part was worth 7 points.

   (a) A saddle-node bifurcation occurs when a pair of fixed points are created.

   - From the picture, this seems to occur when $\lambda_0 = \boxed{-1}$ and $x_0 = -2$.
   - To show it algebraically, we need to check that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$, and $\left.\dfrac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
   - We have $f_{-1}(-2) = -1 - \dfrac{1}{4}(-2)^2 = -2$, $f'_{-1}(-2) = 1$, $f''_{-1}(-2) = -\dfrac{1}{2}$, and $\dfrac{\partial f_\lambda}{\partial \lambda} = 1$ (identically).
   - All of the criteria are satisfied, so there is a saddle-node bifurcation at $\lambda_0 = -1$.

   (b) A period-doubling bifurcation occurs when the fixed-point curve "sprouts" a curve of period-2 points.

   - From the picture, this seems to occur when $\lambda_0 = \boxed{3}$ and $x_0 = 2$.
   - To show it algebraically, we check that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = -1$, and $\left.\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
   - We have $f_3(2) = 3 - \frac{1}{4}(2)^2 = 2$, $f_3'(2) = -1$, and $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda} = \dfrac{1}{4}x \neq 0$, so all of the criteria are satisfied and there is a period-doubling bifurcation at $\lambda_0 = 3$.

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6. Parts (a) and (b) were worth 5 points while part (c) was worth 4 points.

   (a) If $f(x) = -x$, then since f is odd, $f(-x) = -(-x) = x$, so $f(f(x)) = x$. Since $x \neq 0$, $f(x) \neq x$. Thus, x is a point of order 2.

   (b) For $f(x) = x + 4x^5 - 5x^7$, we have $f(0) = 0$ and $f'(0) = 1$ so 0 is neutral. Furthermore, $f''(0) = f^{(3)}(0) = f^{(4)}(0) = 0$ but $f^{(5)}(0) = 480$. So $k = 5$ and the value of the 5th derivative is positive, so by the neutral fixed point theorem, 0 is $\boxed{\text{weakly repelling on both sides}}$.

   (c) Recall that $Sf(x) = \dfrac{f'''(x)}{f'(x)} - \dfrac{3}{2}\left(\dfrac{f''(x)}{f'(x)}\right)^2$. Here, $f' = -2\sin(2x)$, $f'' = -4\cos(2x)$, and $f''' = 8\sin(2x)$, so $Sf(x) = \dfrac{8\sin(2x)}{-2\sin(2x)} - \dfrac{3}{2}\left(\dfrac{-4\cos(2x)}{-2\sin(2x)}\right)^2 = -4 - 6\dfrac{\cos(2x)^2}{\sin(2x)^2} = \boxed{-4 - 6\cot(2x)^2}$.
