Math 3543: Dynamics, Chaos, and Fractals

Midterm 1 (Instructor: Dummit)

February 10th, 2025

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **You are responsible for checking that this exam has all 8 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	14	
2	14	
3	14	
4	10	
5	14	
6	14	
TOTAL	80	

1. (14 points) Let $h(x) = 1 + \frac{5}{6}x^2 + x^3 - \frac{5}{6}x^4$. Here are some values of $h(x)$ and $h'(x)$:

x	-3	-2	-1	0	1	2	3
$h(x)$	-86	-17	0	1	2	-1	-32
$h'(x)$	112	$106/3$	$14/3$	0	$4/3$	$-34/3$	-58

(a) Show that $x = 1$ is a periodic point for h and determine the full cycle it lies in.

(b) Classify the cycle containing 1 as attracting, repelling, or neutral for h.

(c) Prove that $h(x)$ has a fixed point in each of the intervals $(-2, 0)$ and $(1, 2)$.

2. (14 points) Let $D : [0, 1) \to [0, 1)$ be the doubling function $D(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}$.

(Equivalently, $D(x) = 2x$ modulo 1.)

(a) The points $\frac{2}{9}$ and $\frac{1}{15}$ are periodic points for D: find their exact periods.

(b) Suppose n and k are positive integers with $0 < k < 2^n$. Show that $\frac{k}{2^n}$ is an eventually fixed point for D.

(c) Show that D has a periodic cycle of every possible length n. [Hint: Consider $x = \frac{1}{2^n - 1}$.]

3. (14 points) Let $f(x) = \frac{3}{2}x - \frac{1}{2}x^2 - x^3$. Here is a table of some values of f:

x	-3	-2	-1	0	1	2	3
$f(x)$	18	3	-1	0	0	-7	-27

Note also that f has one real-valued 2-cycle, which is approximately $\{-1.7867, 1.4275\}$.

(a) Find the 3 fixed points of f and classify them as attracting, repelling, or neutral.

(b) Find the immediate attracting basin for each attracting fixed point of f.

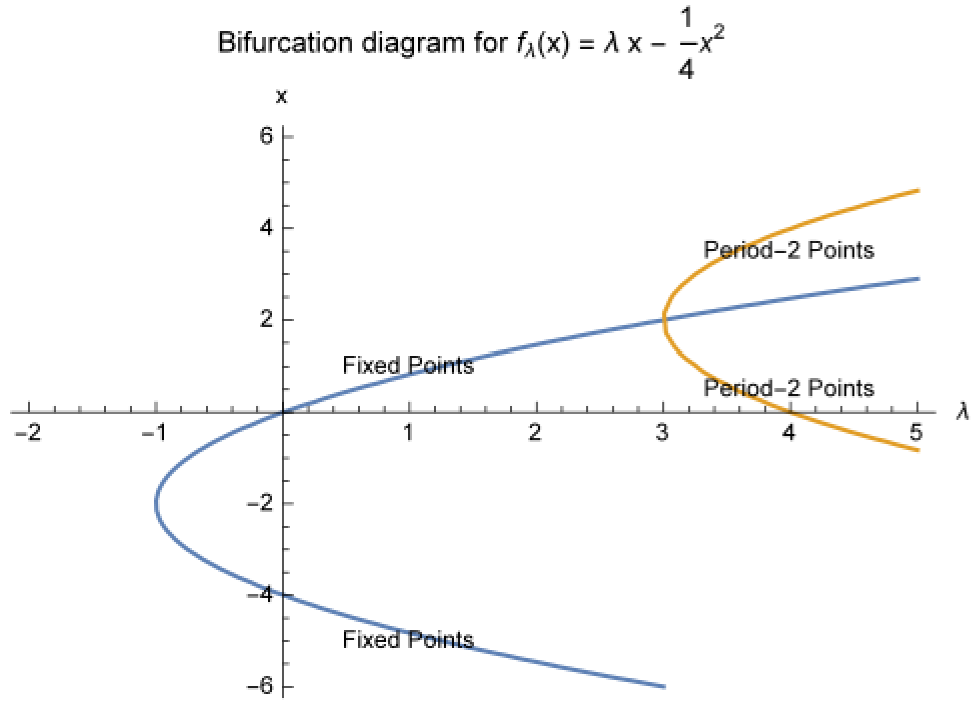
4. (10 points) Let $f(x) = x^3(x-1)$.

(a) Find the Newton iteration function $N(x)$ for f. (No simplification required.)

(b) Find the roots of f and their multiplicities.

(c) To which root of f will the convergence of Newton's method (with a sufficiently close starting value) be faster asymptotically? Why?

5. (14 points) Consider the one-parameter family $f_\lambda(x) = \lambda - \frac{1}{4}x^2$, and notice that $f_\lambda^2(x) = \lambda - \frac{\lambda^2}{4} + \frac{\lambda}{8}x^2 - \frac{1}{64}x^4$. Here is a plot of the bifurcation diagram for this family:



(a) Identify the value of λ_0 where there is a saddle-node bifurcation, and then show algebraically that a saddle-node bifurcation occurs there.

(b) Identify the value of λ_0 where there is a period-doubling bifurcation, and then show algebraically that a period-doubling bifurcation occurs there.

6. (14 points) Solve the following unrelated problems:

(a) Suppose that $f(x)$ is an odd function, so that $f(-x) = -f(x)$ for all x. Show that every nonzero solution to $f(x) = -x$ is a point of exact period 2 for f.

(b) Show that 0 is a neutral fixed point for $f(x) = x + 4x^5 - 5x^7$, and then identify whether it is weakly attracting or weakly repelling for orbits on each side.

(c) For $f(x) = \cos(2x)$, compute the Schwarzian derivative $Sf(x)$.

Blank page for scratch work.