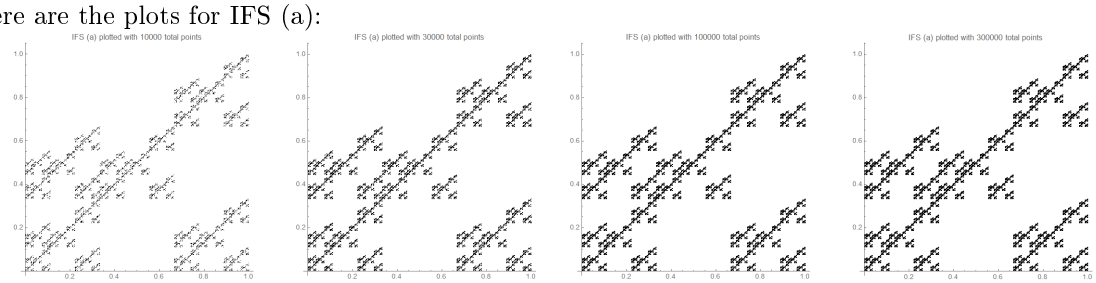
1. For each iterated function system inside the unit square $[0,1]\times[0,1]$, (i) use the chaos game to plot the invariant set with 10000, 30000, 100000, and 300000 total points, and (ii) compute the box-counting dimension for the invariant set to at least 3 decimal places.

   (a) $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x,y) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_4(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_5(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.

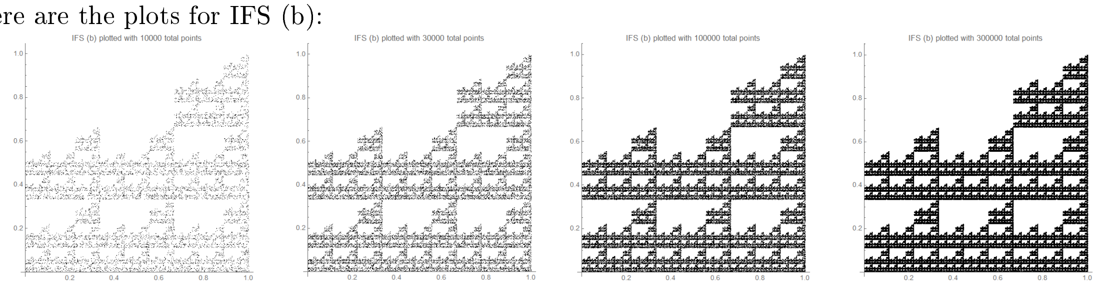
   - Here are the plots for IFS (a):



   - Each similarity has ratio $\frac{1}{3}$, so the Moran equation gives $1 = 5\left(\frac{1}{3}\right)^d$, so $d = \frac{\log 5}{\log 3} = \log_3 5 \approx \boxed{1.46497}$.

   (b) $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_4(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_5(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_6(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_7(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.

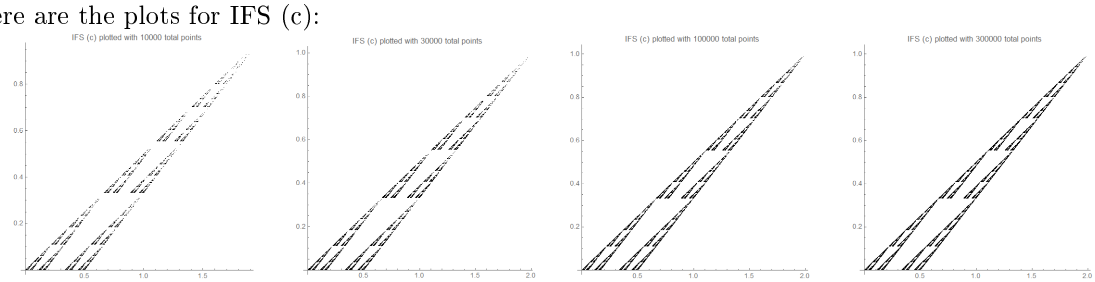
   - Here are the plots for IFS (b):



   - Each similarity has ratio $\frac{1}{3}$, so the Moran equation gives $1 = 7\left(\frac{1}{3}\right)^d$, so $d = \frac{\log 7}{\log 3} = \log_3 7 \approx \boxed{1.77124}$.

   (c) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_3(x) = (\frac{2}{3}x + \frac{1}{3}, \frac{2}{3}y + \frac{1}{3})$.

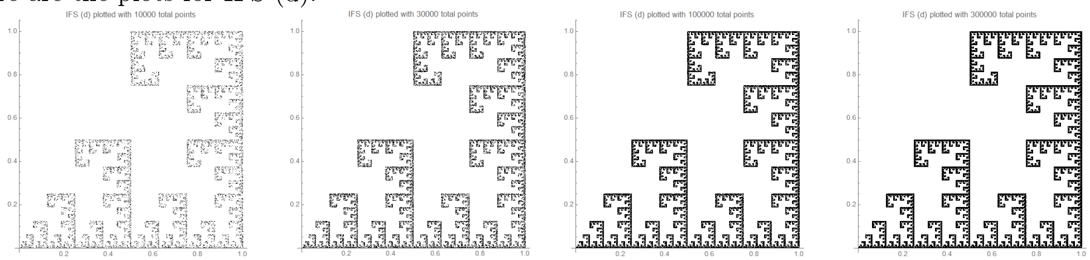
   - Here are the plots for IFS (c):



   - Two similarities have ratio $\frac{1}{3}$ and the last has ratio $\frac{2}{3}$, so the Moran equation gives $1 = 2\left(\frac{1}{3}\right)^d + \left(\frac{2}{3}\right)^d$. This equation cannot be solved algebraically, but using Newton's method gives $d \approx \boxed{1.39495}$.

(d) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{2}x, \frac{1}{2}y)$, $f_2(x) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y)$, $f_3(x) = (1 - \frac{1}{2}y, \frac{1}{2}x + \frac{1}{2})$.

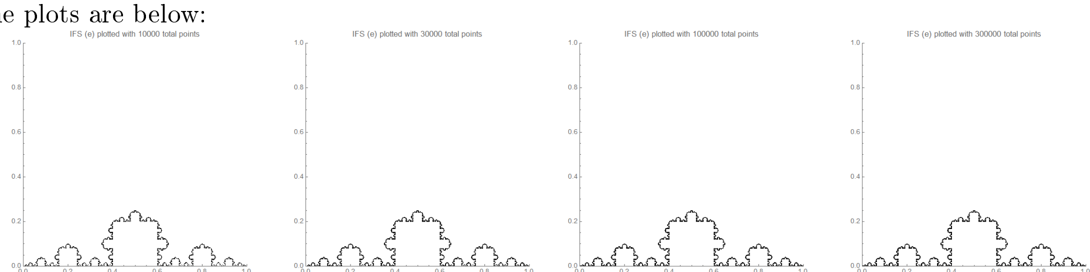
- Here are the plots for IFS (d):



- Each similarity has ratio $\frac{1}{2}$ (the third combines a rotation by $\pi/2$ radians along with the scaling), so the Moran equation gives $1 = 3\left(\frac{1}{2}\right)^d$, so $d = \frac{\log 3}{\log 2} \approx \boxed{1.58496}$.

(e) $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x) = (\frac{2}{5}x, \frac{2}{5}y)$, $f_2(x) = (\frac{2}{5}x + \frac{3}{5}, \frac{2}{5}y)$, $f_3(x) = (\frac{1}{5}x + \frac{2}{5}, \frac{1}{5}y + \frac{2}{5})$, $f_4(x) = (\frac{2}{5} - \frac{1}{5}y, \frac{1}{5}x)$, $f_5(x) = (\frac{1}{5}y + \frac{3}{5}, \frac{1}{5} - \frac{1}{5}x)$.

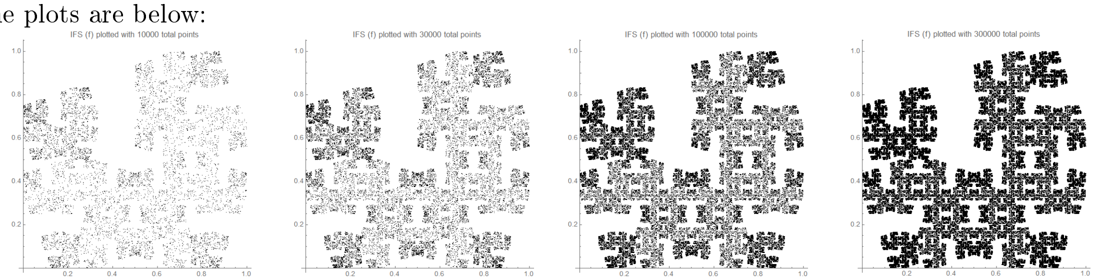
- The plots are below:



- The first two similarities have ratio 2/5 while the last three have ratio 1/5 (the last two also involve rotations) so the Moran equation gives $1 = 2\left(\frac{2}{5}\right)^d + 3\left(\frac{1}{5}\right)^d$. This cannot be solved analytically, but using Newton's method produces $d \approx \boxed{1.28110}$.

(f) $\{f_1, f_2, f_3, f_4\}$, where $f_1(x) = (\frac{1}{2} - \frac{1}{2}y, \frac{1}{2}x)$, $f_2(x) = (\frac{1}{2} + \frac{1}{2}y, \frac{1}{2}x)$, $f_3(x) = (\frac{1}{2} + \frac{1}{2}y, 1 - \frac{1}{2}x)$, $f_4(x) = (\frac{1}{3}x, \frac{1}{2} + \frac{1}{3}y)$.

- The plots are below:



- The first three similarities have ratio 1/2 while the last has ratio 1/3, so the Moran equation gives $1 = 3\left(\frac{1}{2}\right)^d + \left(\frac{1}{3}\right)^d$. This cannot be solved analytically, but using Newton's method produces $d \approx \boxed{1.79992}$.

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2. The goal of this problem is to examine the relationship between the box-counting dimension of a fractal and the number of points needed in the chaos game to draw a reasonably accurate picture of the fractal.

   (a) For each of the iterated function systems you plotted in problem 1, determine the smallest value among those four numbers of points (10000, 30000, 100000, 300000) that produces a reasonably good picture of the fractal. Is there any relation between the number of points needed for a good picture and the box-counting dimension of the set?

   - For system (a) it appears about 100000 points are required (maybe a bit fewer), for (b) 300000 are required (maybe a bit more), for (c) somewhere between 30000 and 100000 are required, for (d) somewhere between 100000 and 300000 are required, and for (e) somewhere between 10000 and 30000 are required, and for (f) more than 300000 are required.
   - The system with the largest box-counting dimension is (f), followed by (b), then by (d), then (a), then (c), then (e). Based on the data, it does seem that an IFS whose box-counting dimension is larger does need more points to make a good plot.

   (b) Suppose we use points of size $\epsilon = 0.001 = 1/1000$ to approximate a fractal of box-counting dimension d. Explain why the number of points required should be roughly 1000^d times a fixed constant factor. (The constant factor comes from the shape of the points; you don't have to explain that part.)

   - The definition of the box-counting dimension says that $d = \lim_{\epsilon\to 0} \dfrac{\ln N(\epsilon)}{\ln(1/\epsilon)}$ where $N(\epsilon)$ is the number of ϵ-boxes (here, points) needed to cover the fractal. If we assume that the limit value is well approximated by the value when $\epsilon = 0.001$, then we obtain the estimate $\ln N(\epsilon) \approx (1/\epsilon)^d = 1000^d$.

   (c) Taking the fixed constant factor to be 2, part (b) says that approximately $2 \cdot 1000^d$ points should be needed to produce a good picture of each of the plots in problem 1. How do those estimates compare to your analysis in part (a)?

   - Based on this heuristic, the sets (a), (b), (c), (d), (e), (f) should require about 50000, 400000, 30000, 110000, 14000, and 500000 points to plot, respectively. These numbers seem to match fairly well the ranges we gave based on the plots.

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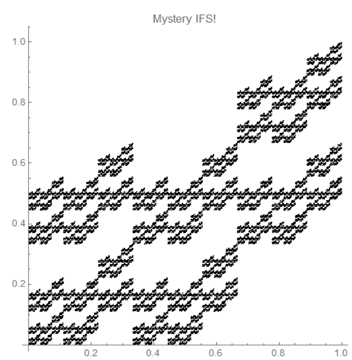
3. Find an iterated function system inside the square $[0,1]\times[0,1]$ whose invariant set is as pictured. [Hint: Break the set into a disjoint union of smaller copies of itself, and then identify the similarities that map the original set onto each smaller copy.]



- We can see six copies of the set contained inside itself: each of them appears to be the same size, a 1/3-scale copy of the original set, with the same orientation.
- The six copies are shifted by, respectively, $(0,0)$, $\left(0,\frac{1}{3}\right)$, $\left(\frac{1}{3},0\right)$, $\left(\frac{1}{3},\frac{1}{3}\right)$, $\left(\frac{2}{3},\frac{1}{3}\right)$, and $\left(\frac{2}{3},\frac{2}{3}\right)$.
- Thus, an iterated function system having this invariant set is $\{f_1, f_2, f_3, f_4, f_5, f_6\}$, where $f_1(x) = \left(\frac{1}{3}x, \frac{1}{3}y\right)$, $f_2(x) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3}\right)$, $f_3(x) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y\right)$, $f_4(x) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3}\right)$, $f_5(x) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3}\right)$, $f_6(x) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right)$.

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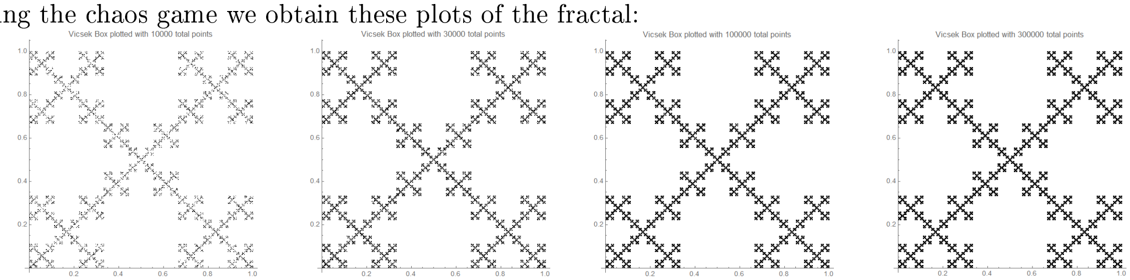
4. Find an iterated function system on the unit square $[0,1] \times [0,1]$ whose invariant set is the Vicsek box fractal described in problem 5 of homework 8. Then use the chaos game to plot the invariant set and compute the box-counting dimension of the fractal to at least 3 decimal places.

   - As described on homework 8, the fractal contains five identical scale-1/3 copies of itself, arranged at the center and corners of the square.
   - So we can take the IFS $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x,y) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3})$, $f_3(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_4(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_5(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.
   - Using the chaos game we obtain these plots of the fractal:



   - Each similarity has ratio $\dfrac{1}{3}$, so the Moran equation gives $1 = 5\left(\dfrac{1}{3}\right)^d$, so $d = \dfrac{\log 5}{\log 3} = \log_3 5 \approx \boxed{1.46497}$.
   - Alternatively, we could use the definition directly: if we cut the plane into squares of side length $\epsilon = 3^{-n}$, then there will be $N(\epsilon) = 5^n$ such squares that contain a point of the box fractal. Then the dimension is the limit $\displaystyle\lim_{n\to\infty} \frac{\log(5^n)}{\log(3^n)} = \lim_{n\to\infty} \frac{n\log(5)}{n\log(3)} = \frac{\log(5)}{\log(3)}$.

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5. Write down three iterated function systems of your choice, and use the chaos game to plot their attracting sets. (Your submission should include the list of functions and the chaos game plot.) Sufficiently interesting results may receive bonus points!

   - This is an open-ended problem. You may see some results of similar IFS creations in problem 1.

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6. The goal of this problem is to analyze a few variations on the center-2/5 Cantor set, in which we start with $[0,1]$ and then at each stage we remove the open middle 2/5 of each interval.

   (a) Find the box-counting dimension of the center-2/5 Cantor set.

   - The center-2/5 Cantor set contains two smaller copies of itself, each at scale 3/10 of the original.
   - By the self-similar dimension theorem, the dimension is the positive real number d satisfying Moran's equation $1 = \left(\dfrac{3}{10}\right)^d + \left(\dfrac{3}{10}\right)^d$. Solving yields $d = \dfrac{\ln(1/2)}{\ln(3/10)} \approx \boxed{0.57572}$.

   Define the left-center-2/5 Cantor set as follows: we begin with $[0,1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and third pieces. Thus, the first iterate consists of the two intervals $[0,1/5] \cup [3/5,1]$.

   (b) Show that the topological dimension of the left-center-2/5 Cantor set is 0. [Hint: Show that between any two points in the set, there is at least one point not in the set.]

   - The total length of the intervals in the nth iterate is $(3/5)^n \to 0$.
   - Thus, if a and b are any two points in the set, there is at least one point x between them that is missing: otherwise the entire interval $[a,b]$ would be in the set, and it has positive length.
   - Now if x is any element of the left-center-2/5 Cantor set that is not 0 or 1, there are elements in the Cantor set arbitrarily close on either side.
   - Thus, by the above, there are also elements not in the Cantor set that are arbitrarily close on either side of x. (This is also true for $x = 0$ and $x = 1$.)

- If we choose such elements c, d with $c < x < d$ and c, d not in the Cantor set, then the open interval (c, d) is a neighborhood of x whose boundary $\{c, d\}$ contains no points from the Cantor set.
- Thus, the Cantor set has topological dimension 0.

(c) Find the box-counting dimension of the left-center-2/5 Cantor set, to at least 3 decimal places.

- The left-center-2/5 Cantor set contains two smaller copies of itself: one at scale 1/5 and another at scale 2/5.
- By the self-similar dimension theorem, the dimension is the positive real number d satisfying Moran's equation $1 = \left(\frac{1}{5}\right)^d + \left(\frac{2}{5}\right)^d$.
- This equation cannot be solved analytically. Using Newton's method, we see that the approximate value is $d \approx \boxed{0.56390}$.

Define the alternating-2/5 Cantor set as follows: we begin with $[0, 1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and fourth pieces. Thus, the first iterate consists of the three intervals $[0, 1/5] \cup [2/5, 3/5] \cup [4/5, 1]$.

(d) Show that the topological dimension of the alternating-2/5 Cantor set is 0.

- The argument is essentially the same: as with the other Cantor set, the total length of the intervals in the nth iterate is $(3/5)^n \to 0$, so between any two points in the set, there are points not in the set.
- Then for any element x in the set, there are elements not in the set arbitrarily close on either side of x. If we pick elements c, d with $c < x < d$ and c, d not in the Cantor set, then the open interval (c, d) is a neighborhood of x whose boundary $\{c, d\}$ contains no points from the Cantor set.
- Thus, the Cantor set has topological dimension 0.

(e) Find the box-counting dimension of the alternating-2/5 Cantor set, to at least 3 decimal places.

- The alternating-2/5 Cantor set contains three smaller copies of itself, each at scale 1/5.
- By the self-similar dimension theorem, the dimension is the positive real number d satisfying Moran's equation $1 = \left(\frac{1}{5}\right)^d + \left(\frac{1}{5}\right)^d + \left(\frac{1}{5}\right)^d$, so that $3 \cdot 5^{-d} = 1$ hence $5^d = 3$ hence $d = \log_5 3 \approx \boxed{0.68261}$.

Remark: You should find in (a), (c), and (e) that changing how we remove the intervals in the construction changes the box-counting dimension of the resulting fractal!
