E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 9, due Wed Apr 2nd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each iterated function system inside the unit square $[0,1]\times[0,1]$, (i) use the chaos game to plot the invariant set with 10000, 30000, 100000, and 300000 total points, and (ii) compute the box-counting dimension for the invariant set to at least 3 decimal places.

   (a) $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x,y) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x,y) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x,y) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_4(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_5(x,y) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.

   (b) $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3})$, $f_3(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_4(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_5(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, $f_6(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3})$, $f_7(x) = (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3})$.

   (c) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{3}x, \frac{1}{3}y)$, $f_2(x) = (\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y)$, $f_3(x) = (\frac{2}{3}x + \frac{1}{3}, \frac{2}{3}y + \frac{1}{3})$.

   (d) $\{f_1, f_2, f_3\}$, where $f_1(x) = (\frac{1}{2}x, \frac{1}{2}y)$, $f_2(x) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y)$, $f_3(x) = (1 - \frac{1}{2}y, \frac{1}{2}x + \frac{1}{2})$.

   (e) $\{f_1, f_2, f_3, f_4, f_5\}$, where $f_1(x) = (\frac{2}{5}x, \frac{2}{5}y)$, $f_2(x) = (\frac{2}{5}x + \frac{3}{5}, \frac{2}{5}y)$, $f_3(x) = (\frac{1}{5}x + \frac{2}{5}, \frac{1}{5}y + \frac{2}{5})$, $f_4(x) = (\frac{2}{5} - \frac{1}{5}y, \frac{1}{5}x)$, $f_5(x) = (\frac{1}{5}y + \frac{3}{5}, \frac{1}{5} - \frac{1}{5}x)$.

   (f) $\{f_1, f_2, f_3, f_4\}$, where $f_1(x) = (\frac{1}{2} - \frac{1}{2}y, \frac{1}{2}x)$, $f_2(x) = (\frac{1}{2} + \frac{1}{2}y, \frac{1}{2}x)$, $f_3(x) = (\frac{1}{2} + \frac{1}{2}y, 1 - \frac{1}{2}x)$, $f_4(x) = (\frac{1}{3}x, \frac{1}{2} + \frac{1}{3}y)$.

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2. The goal of this problem is to examine the relationship between the box-counting dimension of a fractal and the number of points needed in the chaos game to draw a reasonably accurate picture of the fractal.

   (a) For each of the iterated function systems you plotted in problem 1, determine the smallest value among those four numbers of points (10000, 30000, 100000, 300000) that produces a reasonably good picture of the fractal. Is there any relation between the number of points needed for a good picture and the box-counting dimension of the set?

   (b) Suppose we use points of size $\epsilon = 0.001 = 1/1000$ to approximate a fractal of box-counting dimension d. Explain why the number of points required should be roughly 1000^d times a fixed constant factor. (The constant factor comes from the shape of the points; you don't have to explain that part.)

   (c) Taking the fixed constant factor to be 2, part (b) says that approximately $2 \cdot 1000^d$ points should be needed to produce a good picture of each of the plots in problem 1. How do those estimates compare to your analysis in part (a)?

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3. Find an iterated function system on the unit square $[0,1] \times [0,1]$ whose invariant set is the Vicsek box fractal described in problem 5 of homework 8. Then use the chaos game to plot the invariant set and compute the box-counting dimension of the fractal to at least 3 decimal places.

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4. Write down three iterated function systems of your choice, and use the chaos game to plot their attracting sets. (Your submission should include the list of functions and the chaos game plot.) Sufficiently interesting results may receive bonus points!

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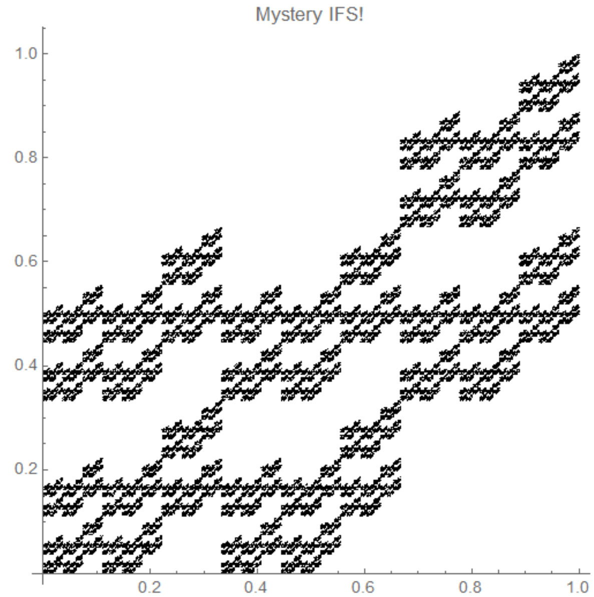
5. Find an iterated function system inside the square $[0,1] \times [0,1]$ whose invariant set is as pictured. [Hint: Break the set into a disjoint union of smaller copies of itself, and then identify the similarities that map the original set onto each smaller copy.]



Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. The goal of this problem is to analyze a few variations on the center-2/5 Cantor set, in which we start with $[0,1]$ and then at each stage we remove the open middle 2/5 of each interval.

   (a) Find the box-counting dimension of the center-2/5 Cantor set.

   Define the left-center-2/5 Cantor set as follows: we begin with $[0,1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and third pieces. Thus, the first iterate consists of the two intervals $[0,1/5] \cup [3/5,1]$.

   (b) Show that the topological dimension of the left-center-2/5 Cantor set is 0. [Hint: Show that between any two points in the set, there is at least one point not in the set.]

   (c) Find the box-counting dimension of the left-center-2/5 Cantor set, to at least 3 decimal places.

   Define the alternating-2/5 Cantor set as follows: we begin with $[0,1]$, and then at each stage, we divide each remaining interval into five equal pieces and remove the open second and fourth pieces. Thus, the first iterate consists of the three intervals $[0,1/5] \cup [2/5,3/5] \cup [4/5,1]$.

   (d) Show that the topological dimension of the alternating-2/5 Cantor set is 0.

   (e) Find the box-counting dimension of the alternating-2/5 Cantor set, to at least 3 decimal places.

   Remark: You should find in (a), (c), and (e) that changing how we remove the intervals in the construction changes the box-counting dimension of the resulting fractal!
