E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 8 Solutions

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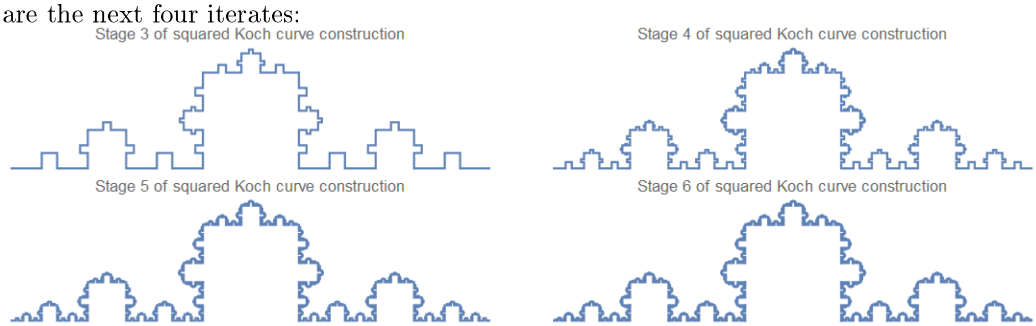
1. The goal of this problem is to study a "squared" version of the Koch curve fractal, constructed as follows: let E_0 be a line segment of length 1. Then, for each $n \geq 1$, define the set E_n to be the set obtained by removing the middle fifth of each segment in E_{n-1} and replacing it with the other three sides of the outwards square sharing those endpoints. The squared Koch curve is the limiting set as $n \to \infty$. The first two iterates are shown below:

   (a) Plot the 3rd, 4th, 5th, and 6th iterates of the construction.

   - Here are the next four iterates:



   (b) Compute the total length of the graph of the nth stage of the construction. What happens as $n \to \infty$?

   - In each iteration, every segment in the graph of length l is replaced by two segments of length $2l/5$ and three segments of length $l/5$, for a total length of $7l/5$. Therefore, the length of the graph is scaled by $7/5$ after each stage.
   - Since the construction starts with a segment of length 1, after the nth stage the total length is $\boxed{(7/5)^n}$, which goes to ∞ as $n \to \infty$.

   (c) Compute the new area created under/inside the graph of the nth stage of the construction. What happens to the total area under/inside the graph as $n \to \infty$? [Hint: The new area produced in each stage is a constant times the area produced in the previous stage.]

   - In the first stage, the only area is a square of side length $1/5$, of area $1/25$.
   - In each subsequent stage, we can see that each area-producing segment from the previous stage of length l (namely, a square of side length l, of area l^2) will sprout two new squares of side length $2l/5$ on the segments adjoining it and three new squares of side length $l/5$ on the segments making it up, for a total new area of $2(2l/5)^2 + 3(l/5)^2 = 11l^2/25$.
   - Thus, the new area produced in this subsequent stage is $11/25$ the area from the previous stage, so at the nth stage, the new area is $\boxed{\frac{1}{25} \cdot (\frac{11}{25})^n}$.
   - Hence the total area as $n \to \infty$ is $\frac{1}{25} + \frac{11}{25} \cdot \frac{1}{25} + (\frac{11}{25})^2 \cdot \frac{1}{25} + \cdots = \frac{1/25}{1 - 11/25} = \boxed{\frac{1}{14}}$.

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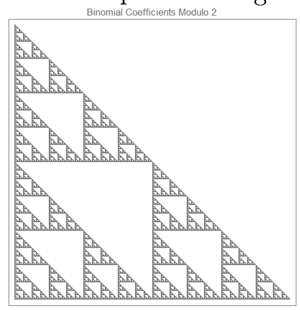
2. The binomial coefficients $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are often displayed in an array called Pascal's triangle.

   (a) Describe the result obtained by (re)drawing the array with a black dot in place of each binomial coefficient that is odd, and with a white dot in place of each binomial coefficient that is even. Can you explain why it has the shape it does?

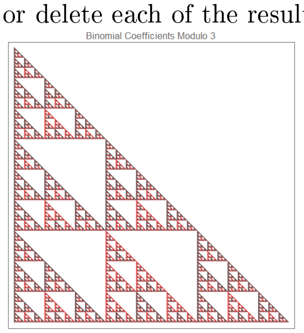
- The result is a right-triangle variant of the Sierpinski triangle:

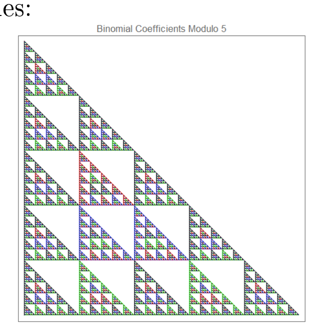


- There are various ways to prove this. One way to see the Sierpinski triangle in the mod 2 case is to use the recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Geometrically, to decide whether a pixel in the nth row of the picture is white or black, we look at the two pixels above: if they are different then the new pixel is black, and if they are the same then the new pixel is white. In particular the leftmost and rightmost pixels are always black.
- The reason we see the triangular patterns emerging is that in rows that are a power of 2, all the pixels are black. (This can be shown by induction.) In the next row, all but the outer pixels will be white – and in each subsequent row, this interval of all-white pixels will shrink by 1 pixel on the left edge until it disappears. There will be smaller regions where clusters of 2^n consecutive white pixels appear that have the same kind of shrinking phenomenon: this accounts for the self-similarity.

(b) What happens if instead you plot the points where the binomial coefficient is congruent to 0 modulo 3? 1 modulo 3? 2 modulo 3? Based on the picture, what is the box-counting dimension of the set where the binomial coefficients are 1 or 2 modulo 3? What do you think will happen with other moduli?

- The results are variants on the Sierpinski triangle construction, where we draw n lines parallel to each side (to divide the triangle into n^2 smaller triangles each similar to the original), and then alternately keep or delete each of the resulting n^2 triangles:





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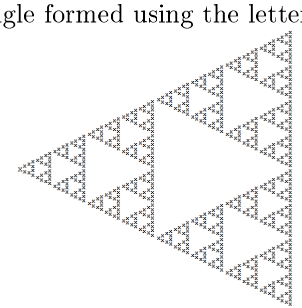
3. If you run the command `Nest[Subsuperscript[#, #, #] &, x, 6]` in Mathematica, a fractal will appear. Which fractal, and why?

- The output produces a Sierpinski triangle formed using the letter x:



- Each iteration of the function turns its input y into y_y^y, so the resulting set will consist of three smaller copies of itself arranged essentially in an equilateral triangle shape. This is precisely the Sierpinski triangle.

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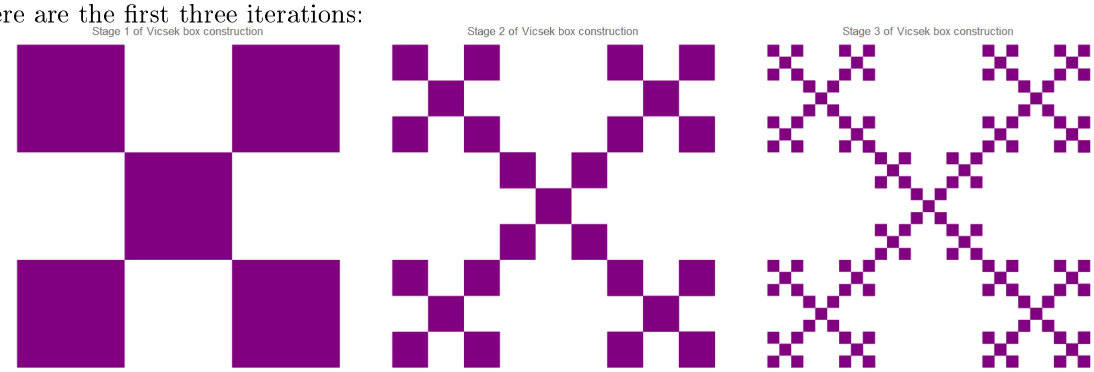
4. The Vicsek box fractal is constructed as follows: begin with a unit square. Then divide it into nine equal subsquares, and then remove the four squares that touch a midpoint of one of the sides. Now apply this procedure to each of the five smaller squares, thus creating 25 smaller squares, and continue iterating. The Vicsek box fractal is the limiting set from this procedure.

(a) Plot the first three iterations of the Vicsek box fractal.

- Here are the first three iterations:



(b) Find the total area and perimeter of the nth iterate and determine what happens to each as $n \to \infty$.

- At each stage, we get five new squares each of side length $1/3$ the previous square. So the total area scales by a factor of $5/9$ and the total perimeter scales by a factor of $5/3$ from each stage to the next.
- By a trivial induction, the nth-stage area is $(5/9)^n$ which clearly goes to zero as $n \to \infty$, and the nth-stage perimeter is $4(5/3)^n$ which clearly goes to ∞ as $n \to \infty$.

(c) Show that the topological dimension of the Vicsek box fractal is 1. [Hint: Show that the box fractal contains a line, and also that, for any of the individual squares in the nth iterate, there is a circle that passes through its vertices but no other points in the set.]

- First observe that the topological dimension is not zero. This follows for example by noticing that it contains the diagonal of the starting square, which is a set of topological dimension 1.
- Now suppose x is a point in the box fractal. We will show that there is an arbitrarily small neighborhood of x whose boundary intersects the box fractal in a finite set of points.
- Consider the square S containing x in the nth iterate of the construction: we claim that the circle C passing through the four vertices of this square does not contain any other points from the nth iterate. This is obvious from a picture, but explicitly: the only points lying inside C are the points in the square S_n along with some points the four equally-sized squares touching S_n along its edges, but none of these squares lies in the nth iterate of the construction. The only point lying on C are therefore the vertices of the square S_n.
- The interior of C is then a neighborhood of x whose boundary intersects the box fractal in a set of topological dimension zero. Since we can choose this circle to be arbitrarily small (as its radius is $\sqrt{2}/3^n$), we are done.

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5. Let $N(x) = \dfrac{x^2-1}{2x}$ for $x \neq 0$. The goal of this problem is to characterize the set of points S where all the iterates of N are defined, and then to prove that N is chaotic on this set.

(a) Show that $N(x)$ is a Newton iterating function for a polynomial $p(x)$ that has no real roots. Conclude that N has no attracting fixed points.

- Since $N(x) = x - \dfrac{x^2+1}{2x}$ we see that N is the Newton iterating function for $p(x) = x^2 + 1$, which has no real roots.
- From Newton's fixed point theorem we therefore see that N has no attracting fixed points. (We would thus expect N to behave in an unpredictable manner.)

(b) Show that $h(t) = \cot(\pi t)$ is a homeomorphism from $(0, 1)$ to $\mathbb{R}$.

- Clearly h is continuous since it is differentiable, it is one-to-one since its derivative is always negative, and it is surjective since $\cot(t) \to \infty$ as $t \to 0^+$ and $\cot(t) \to -\infty$ as $t \to \pi^-$.
- Also, $h^{-1}(t) = \frac{1}{\pi}\cot^{-1}(t)$ is differentiable on $\mathbb{R}$, so it is also continuous.

(c) If D is the doubling map restricted to $(0,1)$, show that $h : (0,1) \to \mathbb{R}$ satisfies the relation $h(D(x)) = N(h(x))$ for all $x \neq 1/2$.

- We have $N(h(x)) = \dfrac{\cot^2(\pi x) - 1}{2\cot(\pi x)} = \cot(2\pi x)$ by the cotangent double-angle formula $\cot(2t) = \dfrac{\cot^2 t - 1}{2\cot t}$.
- For $x \in (0, 1/2)$ we have $h(D(x)) = h(2x) = \cot(2\pi x) = N(h(x))$ as required.
- Also, for $x \in (1/2, 1)$ we have $h(D(x)) = h(2x-1) = \cot(2\pi x - \pi) = \cot(2\pi x) = N(h(x))$ since cotangent has period π.

(d) Show that if $x_0 = \cot(\pi r_0)$ then $N^n(x_0) = \cot(2^n \pi r_0)$.

- From the calculations in part (c) we see that if $x_0 = \cot(\pi r_0)$ then $N(x_0) = \cot(2\pi r_0)$.
- Iterating this n times yields $N^n(x_0) = \cot(2^n \pi r_0)$.

(e) Show that the set of points S where all iterates of N are defined is the set of points not of the form $\cot(k\pi/2^n)$ for any integers k and n.

- Motivated by part (d), we will show more specifically that the set of points S_n where N^n is undefined is the set of points of the form $\cot(k\pi/2^n)$.
- The central observation is that $\cot(t)$ is undefined precisely when t is an integral multiple of π.
- So now suppose $x_0 = \cot(\pi r_0)$ and that $N^n(x_0)$ is undefined.
- By part (d), since $N^n(x_0) = \cot(2^n \pi r_0)$, we see that this expression is undefined precisely when $2^n \pi r_0$ is an integral multiple of π: that is, when $2^n \pi r_0 = \pi k$ for some integer k: thus, $r_0 = k/2^n$, and so $x_0 = \cot(k\pi/2^n)$.

(f) If T is the set of points in $(0,1)$ not having a terminating base-2 decimal expansion (i.e., not of the form $k/2^n$ for any integers k and n), show that $h(t) = \cot(\pi t)$ yields an equivalence between the dynamical systems (T, D) and (S, N).

- This essentially follows from what we have already done in parts (b), (c), and (e).
- We do need to verify that $h(t) = \cot(\pi t)$ remains a bijection when we restrict it from $(0,1) \to \mathbb{R}$ to $T \to S$, since we already know that h and h^{-1} are continuous from part (b).
- The fact that h is a bijection from T to S follows from part (e): the points in T are the elements in $(0,1)$ that are not of the form $k/2^n$ for any integers k, n, and the points in S are the elements of $\mathbb{R}$ that are not of the form $\cot(k\pi/2^n)$ for any integers k, n.
- So $h : S \to T$ is a homeomorphism. It still obeys the conjugacy relation as we saw in part (c), so it is a conjugacy between (T, D) and (S, N) as claimed.

(g) Conclude that N is chaotic on S.

- We know that the doubling map is chaotic on $[0, 1)$, so it is also chaotic on any smaller set. If we restrict D to the set T, then (D, T) is chaotic, and D is also continuous on T (since we have removed the discontinuity at $x = 1/2$, as that point is not in T).
- Since N is also continuous on S, $h : S \to T$ is a continuous surjection, and S is obviously infinite, our theorem on conjugacy and chaotic systems implies that (S, N) is chaotic.

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6. Let the tent map be $T(x) = \begin{cases} 2x & \text{for } 0 \le x < 1/2 \\ 2 - 2x & \text{for } 1/2 \le x \le 1 \end{cases}$, and then, for $0 \le h \le 1$, define the truncated tent map to be $T_h(x) = \min(h, T(x))$. The goal of this problem is to explore how these maps can be used to prove the converse of Sarkovskii's theorem.

(a) Find the 2-cycles, 3-cycles, 4-cycles, and 5-cycles for the map T. (There are 1, 2, 3, and 6 respectively.)

- We can rapidly compute these using Mathematica's Reduce command.

- The 2-cycle is $\{\frac{2}{5}, \frac{4}{5}\}$ and the 3-cycles are $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$ and $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$.
- The 4-cycles are $\{\frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}\}$, $\{\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{14}{15}\}$, $\{\frac{6}{17}, \frac{12}{17}, \frac{10}{17}, \frac{14}{17}\}$.
- The 5-cycles are $\{\frac{2}{33}, \frac{4}{33}, \frac{8}{33}, \frac{16}{33}, \frac{32}{33}\}$, $\{\frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}, \frac{30}{31}\}$, $\{\frac{2}{11}, \frac{4}{11}, \frac{8}{11}, \frac{6}{11}, \frac{10}{11}\}$, $\{\frac{6}{31}, \frac{12}{31}, \frac{24}{31}, \frac{14}{31}, \frac{28}{31}\}$, $\{\frac{10}{33}, \frac{20}{33}, \frac{26}{33}, \frac{14}{33}, \frac{28}{33}\}$, and $\{\frac{10}{31}, \frac{20}{31}, \frac{22}{31}, \frac{18}{31}, \frac{26}{31}\}$.

(b) Suppose $0 < h \le 1$ and $m \ge 1$. Show that any m-cycle $\{x_1, x_2, \cdots, x_m\}$ for T_h is also an m-cycle for T, except possibly if some $x_i = h$. [Hint: $T(x) = T_h(x)$ whenever $T(x) \le h$.]

- The point is that T and T_h agree everywhere except for the points where $T(x) > h$. If $T(x_i) \le h$ for all x_i then $T(x_i) = T_h(x_i)$ so $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T.
- If $T(x_i) > h$ for some x_i, then $T_h(x_i) = h$, but this would mean $x_{i+1} = h$ and thus that h is contained in the m-cycle.

(c) Suppose that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T, and $\alpha = \max(x_1, x_2, \ldots, x_m)$. Show that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T_h for all $\alpha \le h \le 1$, but is not an m-cycle for T_h when $h < \alpha$. [Hint: $T_h(x) = T_\alpha(x)$ whenever $x \le \alpha$; for the second part, consider the range of T_h.]

- Because $T_h(x) = T(x)$ whenever $T(x) \le h$, we see that $T_\alpha(x_i) = T(x_i) = T_h(x_i)$ for each i since $T_\alpha(x_i) \le \alpha \le h$ by definition of α and h.
- So $\{x_1, \ldots, x_m\}$ is an m-cycle for T_α.
- For the last part, the range of T_h is $[0, h]$, so if $x_i > h$ then x_i cannot be in the range of T_h, making it impossible for $\{x_1, x_2, \cdots, x_m\}$ to be an m-cycle for T_h.

(d) Show that the map $T_{4/5}$ has cycles of lengths 2 and 1 but no others.

- By part (c), it has a 2-cycle since $\{\frac{2}{5}, \frac{4}{5}\}$ is a 2-cycle for T as we saw in part (a). It also has a fixed point $x = 0$.
- Furthermore, also by part (c), since $\frac{4}{5}$ is less than $\frac{16}{17}$, $\frac{14}{15}$, and $\frac{14}{17}$, none of the 4-cycles for T is a 4-cycle for $T_{4/5}$.
- By part (b), or part (c), we therefore see that $T_{4/5}$ has no 4-cycles. Then by the forward direction of Sarkovskii's theorem, $T_{4/5}$ cannot have an m-cycle for any m that appears before 4 in the Sarkovskii ordering: thus, it can only have cycles of length 2 and 1.

(e) Show that the map $T_{28/33}$ has cycles of every length except 3.

- By part (c), it has a 5-cycle since $\{\frac{10}{33}, \frac{20}{33}, \frac{26}{33}, \frac{14}{33}, \frac{28}{33}\}$ is a 5-cycle for T as we saw in part (a).
- Furthermore, also by part (c), since $\frac{28}{33}$ is less than $\frac{8}{9}$ and $\frac{6}{7}$, neither of the 3-cycles for T is a 3-cycle for $T_{28/33}$.
- By part (b) we therefore see that $T_{28/33}$ has no 3-cycles. Then by the forward direction of Sarkovskii's theorem, $T_{28/33}$ has cycles of every length except 3.

(f) Show that the map $T_{106/127}$ has cycles of every length except 3 and 5.

- By part (c), it has a 7-cycle, since some short calculation shows that $\{\frac{106}{127}, \frac{42}{127}, \frac{84}{127}, \frac{86}{127}, \frac{82}{127}, \frac{90}{127}, \frac{74}{127}\}$ is a 7-cycle for T.
- Furthermore, also by part (c), since $\frac{106}{127}$ is less than each of $\frac{32}{33}$, $\frac{30}{31}$, $\frac{10}{11}$, $\frac{28}{31}$, $\frac{28}{33}$, $\frac{26}{31}$, none of the 5-cycles for T is a 5-cycle for $T_{28/33}$.
- By part (b) we therefore see that $T_{28/33}$ has no 5-cycles. Then by the forward direction of Sarkovskii's theorem, $T_{28/33}$ has cycles of every length except 3 and 5. (It cannot have any 3-cycles since having one would force a 5-cycle.)
