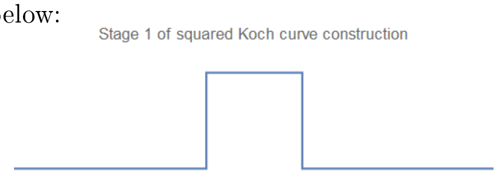
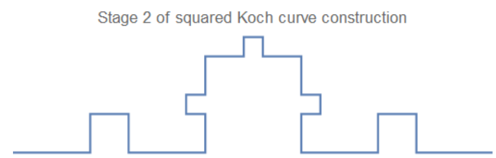
E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 8, due Wed Mar 19th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. The goal of this problem is to study a "squared" version of the Koch curve fractal, constructed as follows: let E_0 be a line segment of length 1. Then, for each $n \geq 1$, define the set E_n to be the set obtained by removing the middle fifth of each segment in E_{n-1} and replacing it with the other three sides of the outwards square sharing those endpoints. The squared Koch curve is the limiting set as $n \to \infty$. The first two iterates are shown below:





   (a) Plot the 3rd, 4th, 5th, and 6th iterates of the construction.

   (b) Compute the total length of the graph of the nth stage of the construction. What happens as $n \to \infty$?

   (c) Compute the new area created under/inside the graph of the nth stage of the construction. What happens to the total area under/inside the graph as $n \to \infty$? [Hint: The new area produced in each stage is a constant times the area produced in the previous stage.]

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2. The binomial coefficients $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ are often displayed in an array called Pascal's triangle.

   (a) Describe the result obtained by (re)drawing the array with a black dot in place of each binomial coefficient that is odd, and with a white dot in place of each binomial coefficient that is even. Can you explain why it has the shape it does?

   (b) What happens if instead you plot the points where the binomial coefficient is congruent to 0 modulo 3? 1 modulo 3? 2 modulo 3? Based on the picture, what is the box-counting dimension of the set where the binomial coefficients are 1 or 2 modulo 3? What do you think will happen with other moduli?

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3. If you run the command `Nest[Subsuperscript[#, #, #] &, x, 6]` in Mathematica, a fractal will appear. Which fractal, and why?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. The Vicsek box fractal is constructed as follows: begin with a unit square. Then divide it into nine equal subsquares, and then remove the four squares that touch a midpoint of one of the sides. Now apply this procedure to each of the five smaller squares, thus creating 25 smaller squares, and continue iterating. The Vicsek box fractal is the limiting set from this procedure.

   (a) Plot the first three iterations of the Vicsek box fractal.

   (b) Find the total area and perimeter of the nth iterate and determine what happens to each as $n \to \infty$.

   (c) Show that the topological dimension of the Vicsek box fractal is 1. [Hint: Show that the box fractal contains a line, and also that, for any of the individual squares in the nth iterate, there is a circle that passes through its vertices but no other points in the set.]

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5. Let $N(x) = \dfrac{x^2 - 1}{2x}$ for $x \neq 0$. The goal of this problem is to characterize the set of points S where all the iterates of N are defined, and then to prove that N is chaotic on this set.

(a) Show that $N(x)$ is a Newton iterating function for a polynomial $p(x)$ that has no real roots. Conclude that N has no attracting fixed points.

(b) Show that $h(t) = \cot(\pi t)$ is a homeomorphism from $(0, 1)$ to $\mathbb{R}$.

(c) If D is the doubling map restricted to $(0, 1)$, show that $h : (0, 1) \to \mathbb{R}$ satisfies the relation $h(D(x)) = N(h(x))$ for all $x \neq 1/2$.

(d) Show that if $x_0 = \cot(\pi r_0)$ then $N^n(x_0) = \cot(2^n \pi r_0)$.

(e) Show that the set of points S where all iterates of N are defined is the set of points not of the form $\cot(k\pi/2^n)$ for any integers k and n.

(f) If T is the set of points in $(0, 1)$ not having a terminating base-2 decimal expansion (i.e., not of the form $k/2^n$ for any integers k and n), show that $h(t) = \cot(\pi t)$ yields an equivalence between the dynamical systems (T, D) and (S, N).

(g) Conclude that N is chaotic on S.

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6. Let the tent map be $T(x) = \begin{cases} 2x & \text{for } 0 \le x < 1/2 \\ 2 - 2x & \text{for } 1/2 \le x \le 1 \end{cases}$, and then, for $0 \le h \le 1$, define the truncated tent map to be $T_h(x) = \min(h, T(x))$. The goal of this problem is to explore how these maps can be used to prove the converse of Sarkovskii's theorem.

(a) Find the 2-cycles, 3-cycles, 4-cycles, and 5-cycles for the map T. (There are 1, 2, 3, and 6 respectively.)

(b) Suppose $0 < h \le 1$ and $m \ge 1$. Show that any m-cycle $\{x_1, x_2, \cdots, x_m\}$ for T_h is also an m-cycle for T, except possibly if some $x_i = h$. [Hint: $T(x) = T_h(x)$ whenever $T(x) \le h$.]

(c) Suppose that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T, and $\alpha = \max(x_1, x_2, \ldots, x_m)$. Show that $\{x_1, x_2, \cdots, x_m\}$ is an m-cycle for T_h for all $\alpha \le h \le 1$, but is not an m-cycle for T_h when $h < \alpha$. [Hint: $T_h(x) = T_\alpha(x)$ whenever $x \le \alpha$; for the second part, consider the range of T_h.]

(d) Show that the map $T_{4/5}$ has cycles of lengths 2 and 1 but no others.

(e) Show that the map $T_{28/33}$ has cycles of every length except 3.

(f) Show that the map $T_{106/127}$ has cycles of every length except 3 and 5.
