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1. For the following dynamical systems (X, f), does f have sensitive dependence on all of X? If not, does it have sensitive dependence at any individual points in X? (Give some justification.)

   (a) $X = [0,1]$, $f(x) = x^3$.

   - This map $\boxed{\text{does not have sensitive dependence on } X}$, because all points in $[0,1)$ are attracted to the attracting fixed point at $x = 0$.
   - It $\boxed{\text{does have sensitive dependence at } x = 1}$, since points with $x < 1$ will be attracted to 0 but the orbit of 1 remains fixed at 1.

   (b) $X = \mathbb{R}$, $f(x) = 2x + 3$.

   - This map $\boxed{\text{does have sensitive dependence on } X}$: for any points x and y we have $|f(x) - f(y)| = 2|x - y|$, so the distance between any two points doubles with each application of f.

   (c) $X = \mathbb{R}$, $f(x) = \cos(x)$.

   - This map $\boxed{\text{does not have sensitive dependence on } X}$, or at any point of X, because the attracting basin of the attracting fixed point at $x \approx 0.739$ is $(-\infty, \infty)$.
   - Every orbit will be (very rapidly) attracted to this attracting fixed point, so there is no dependence at all on the starting value.

   (d) $X = \Sigma_2$, $f(d_0 d_1 d_2 d_3 d_4 d_5 \cdots) = (d_2 d_3 d_4 d_5 \cdots)$.

   - This map $\boxed{\text{does have sensitive dependence on } X}$: it is the second iterate of the shift map, and essentially the same argument as used for the shift map will demonstrate that it has sensitive dependence with $\beta = 1/2$.
   - Explicitly, suppose $x = (d_0 d_1 d_2 d_3 \cdots)$ and $\epsilon > 0$ are given, and choose n such that $2^{-2n} < \epsilon$.
   - Then choose $y = (d_0 d_1 \cdots d_{2n-1} e_{2n} 000 \cdots)$ where $e_{2n} = 1 - d_{2n}$. Since x and y agree in the 0th through $(2n-1)$st coordinates, $d(x, y) \le 2^{-2n} < \epsilon$.
   - Also, $f^n(x) = (d_{2n} d_{2n+1} d_{2n+2} \cdots)$ while $f^n(y) = (e_{2n} 00 \cdots)$ so $d(f^n(x), f^n(y)) \ge 1$.
   - Thus the orbits of x and y eventually move a distance at least 1 away from each other, so f has sensitive dependence as claimed.

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2. Find exact expressions for the three 4-cycles of $q_{-2}(x) = x^2 - 2$.

   - From in class, we know that $q_{-2}(2\cos t) = 2\cos 2t$, and the points of period dividing n for q_{-2} are the 2^{n-1} values $x = 2\cos\frac{2k\pi}{2^n+1}$ for $1 \le k \le 2^{n-1}$ along with the 2^{n-1} values $x = 2\cos\frac{2k\pi}{2^n-1}$ for $0 \le k \le 2^{n-1} - 1$.
   - So the points of period dividing 2 are $2\cos\frac{2\pi}{5}$, $2\cos\frac{4\pi}{5}$, $2\cos 0$, $2\cos\frac{2\pi}{3}$.
   - Then the points of period dividing 4 are $2\cos\frac{2\pi}{15}$, $2\cos\frac{4\pi}{15}$, $2\cos\frac{6\pi}{15} = 2\cos\frac{2\pi}{5}$, $2\cos\frac{8\pi}{15}$, $2\cos\frac{10\pi}{15} = 2\cos\frac{2\pi}{3}$, $2\cos\frac{12\pi}{15} = 2\cos\frac{4\pi}{5}$, $2\cos\frac{14\pi}{15}$, along with $2\cos 0$, $2\cos\frac{2\pi}{17}$, $2\cos\frac{4\pi}{17}$, $2\cos\frac{6\pi}{17}$, $2\cos\frac{8\pi}{17}$, $2\cos\frac{10\pi}{17}$, $2\cos\frac{12\pi}{17}$, $2\cos\frac{14\pi}{17}$, $2\cos\frac{16\pi}{17}$.
   - As cycles these are $\boxed{2\cos\frac{2\pi}{15} \to 2\cos\frac{4\pi}{15} \to 2\cos\frac{8\pi}{15} \to 2\cos\frac{14\pi}{15}}$, $\boxed{2\cos\frac{2\pi}{17} \to 2\cos\frac{4\pi}{17} \to 2\cos\frac{8\pi}{17} \to 2\cos\frac{16\pi}{17}}$, and $\boxed{2\cos\frac{6\pi}{17} \to 2\cos\frac{12\pi}{17} \to 2\cos\frac{10\pi}{17} \to 2\cos\frac{14\pi}{17}}$.

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3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Answer the following:

(a) If f has a point of exact period 11, does it necessarily have a point of exact period 32? 26? 17? 9? 4?

- By Sarkovskii's theorem and its converse, we need only determine which of these integers appear after 11 in the Sarkovskii ordering $3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots \rhd 4 \rhd 2 \rhd 1$.
- The only integers appearing before 11 are 3, 5, 7, and 9. So it does have points of period $\boxed{32,\ 26,\ 17,\ 4}$ but not necessarily 9.

(b) If f has a point of exact period 22, does it necessarily have a point of exact period 32? 26? 17? 9? 4?

- Again by Sarkovskii's theorem and its converse, we need only determine which of these integers appear after 26 in the Sarkovskii ordering.
- All the odd integers along with 6, 10, 14 appear before 22. So it does have points of period $\boxed{32,\ 26,\ 4}$ but not necessarily 9 or 17.

(c) If f has a point of exact period 16, does it necessarily have a point of exact period 32? 26? 17? 9? 4?

- Again, we need only determine which of these integers appear after 16 in the Sarkovskii ordering.
- The only integers after 16 are 8, 4, 2, and 1, so f is only required to have a point of period $\boxed{4}$ but not any of the others.

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4. Recall the definition of the tripling map $T(x) = \begin{cases} 3x & \text{for } 0 \le x < 1/3 \\ 3x - 1 & \text{for } 1/3 \le x < 2/3 \\ 3x - 2 & \text{for } 2/3 \le x < 1 \end{cases}$ on the interval $[0, 1)$. Equivalently, $T(x) = 3x$ modulo 1. The goal of this problem is to prove that T is chaotic.

(a) Suppose $\alpha = 0.d_1d_2d_3d_4\ldots$ in base 3, where, if α has a terminating expansion, we use that expansion instead of the non-terminating one. Show that $T(\alpha) = 0.d_2d_3d_4\ldots$.

- If $\alpha = 0.d_1d_2d_3d_4\ldots$, then $3\alpha = d_1.d_2d_3d_4\ldots$, so 3α is congruent to $0.d_2d_3d_4\ldots$ mod 1, and $0.d_2d_3d_4\ldots$ lies in the interval $[0, 1]$.
- The only potential ambiguity is if α has two base-3 representations where the digit d_1 can have two different possible values.
- There are only three such α: $\alpha = 0.1 = 0.0\overline{2}$, $\alpha = 0.2 = 0.1\overline{2}$, and $\alpha = 1.0 = 0.\overline{2}$, and it is straightforward to check that the statement also holds for these three values. So $T(\alpha) = 0.d_2d_3d_4\ldots$ in all cases.

(b) Show that the periodic points for T in $[0, 1)$ are precisely the points with a periodic base-3 decimal expansion. Conclude that the set of periodic points for T is dense in $[0, 1)$.

- Suppose that $T^n(\alpha) = \alpha$, where $\alpha = 0.d_1d_2d_3\ldots$ is a non-terminating base-3 expansion.
- By part (a), we know that $T^n(\alpha) = 0.d_{n+1}d_{n+2}d_{n+3}\ldots$, meaning that $d_i = d_{n+i}$ for each $i \ge 1$.
- In other words, the base-3 expansion of α repeats every n digits.
- Conversely, it is immediate that if the base-3 expansion of α repeats every n digits, and if α does not have a terminating base-3 expansion, then $T^n(\alpha) = \alpha$.
- If $\alpha \in [0, 1)$ does have a terminating base-3 expansion, then it cannot be periodic unless $\alpha = 0$ (since the digits in the expansion are eventually all zeroes), and 0 is a fixed point of T.
- For the last statement, observe that if $x = 0.x_1x_2x_3x_4\ldots$, then there is a sequence of periodic points converging to x, namely: $0.\overline{x_1}$, $0.\overline{x_1x_2}$, $0.\overline{x_1x_2x_3}$, $0.\overline{x_1x_2x_3x_4}$, ...

(c) Let $\gamma = 0.\underbrace{012}_{\text{length 1}}\ \underbrace{00\,01\,02\,10\cdots 22}_{\text{length 2}}\cdots$ be the real number obtained by listing all length-1 sequences in base 3, then all length-2 sequences, then all length-3 sequences, and so forth. Show that the orbit of γ is dense in $[0,1)$, and conclude T is transitive on $[0,1)$.

- By part (a), $T^k(\gamma)$ deletes the first k digits of the base-3 decimal expansion of γ.
- So in particular, for any sequence of digits, there is some k such that the decimal expansion of $T^k(\gamma)$ that begins with that sequence of digits.
- Thus for any $x \in [0,1)$ and any $n \geq 1$, there is an element x_n in the orbit of γ under T that agrees with x to n decimal places.
- Then $\lim_{n\to\infty} x_n = x$, since $|x_n - x| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} \leq 3^{-n}$.
- Since there is a sequence of points in the orbit of γ converging to x for any $x \in [0,1)$, this means γ has a dense orbit, and therefore in particular T is transitive on $[0,1)$.

(d) Suppose x and y are in $[0,1)$ with $y > x$. If both x and y lie in the same interval $[0,1/3)$, $[1/3,2/3)$, or $[2/3,1)$, show that $|T(y) - T(x)| = 3\,|y-x|$. If they lie in different intervals, show that at least one of $|y-x|$ and $|T(y)-T(x)|$ must be $\geq 1/4$.

- If x and y are both in $[0,1/3)$, $[1/3,2/3)$, or $[2/3,1)$ then $T(y) = 3y - a$ and $T(x) = 3x - a$ for $a \in \{0,1,2\}$ so $|T(y) - T(x)| = 3\,|y-x|$ in each case.
- Now suppose y and x lie in different intervals. If $y - x \geq 1/4$ then we are already done.
- Otherwise, assume $y - x < 1/4$. If $x \in [0,1/3)$ then $y \in [1/3,2/3)$, so $T(y) = 3y-1$ and $T(x) = 3x$. Then $|T(y) - T(x)| = |1 - 3(y-x)| > 1/4$, as required.
- If $x \in [1/3,2/3)$ then $y \in [2/3,1)$ so $T(y) = 3y - 2$ and $T(x) = 3x - 1$. then $|T(y) - T(x)| = |1 - 3(y-x)| > 1/4$ once again.

(e) Suppose x and y are in $[0,1)$ and that $y \neq x$. Show that there is some $k \geq 0$ for which $\left|T^k(y) - T^k(x)\right| \geq 1/4$. Conclude that T has sensitive dependence on $[0,1)$.

- Let $n \geq 1$ and consider the iterates $T^n(x)$ and $T^n(y)$. If they lie in different intervals $[0,1/3)$, $[1/3,2/3)$, $[2/3,1)$, then by part (d) then it is either true that $|T^n(x) - T^n(y)| \geq 1/4$ or $\left|T^{n+1}(x) - T^{n+1}(y)\right| \geq 1/4$, so we are done.
- We only need to eliminate the possibility that $T^n(x)$ and $T^n(y)$ always lie in the same interval. By part (d), if this is true then $|T^n(x) - T^n(y)| = 3^n\,|x-y|$. However, because $x \neq y$, eventually $3^n\,|x-y|$ will exceed $1/3$: but this is impossible by the assumption that $T^n(x)$ and $T^n(y)$ both lie in an interval of length $1/3$.
- Finally, if we take any $\beta < 1/4$, then our results show that the definition of sensitive dependence is satisfied for that value of β.

(f) Show that T is chaotic on $[0,1)$.

- By parts (b), (c), and (e), T has a dense set of periodic points, T is transitive, and T has sensitive dependence. Therefore, by definition, T is chaotic.

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5. Let $g : [0,2] \to [0,2]$ be defined via $g(x) = \begin{cases} x^2 - 3x + 2 & \text{for } 0 \leq x \leq 1 \\ x - 1 & \text{for } 1 < x \leq 2 \end{cases}$. Show that g has a point of exact period n for each $n \geq 1$.

- Since g is continuous (as both component functions are continuous and they are both equal to 0 at $x = 1$), by Sarkovskii's theorem the statement that g has a point of exact period n for each $n \geq 1$ is equivalent to the statement that g has a point of order 3.
- So we look for a 3-cycle. A tiny amount of searching reveals the 3-cycle $\{0,2,1\}$, as $g(0) = 2$, $g(2) = 1$, and $g(1) = 0$. Thus by Sarkovskii's theorem, g has a point of exact period n for each $n \geq 1$.

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6. Suppose $f : I \to I$ is a continuous function on a closed interval $I \subseteq \mathbb{R}$.

(a) Suppose that the set of fixed points of f is dense in I. Prove that f must be the identity function (i.e., that $f(x) = x$ for all $x \in I$). [Hint: If $\lim_{n\to\infty} a_n = a$ is any convergent sequence, then $\lim_{n\to\infty} f(a_n) = f(a)$.]

- Let $a \in I$. By the assumption that the set of fixed points is dense, there is some sequence of fixed points $\{a_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} a_n = a$.
- Then since f is continuous and $f(a_n) = a_n$ for all n, we see that $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(a_n) = f(a)$.
- So $f(a) = a$. This is true for any $a \in I$, so f is the identity function.

(b) Suppose that the set of periodic points of period $\leq n$ for f is dense in I. Prove that some iterate of f must be the identity function. [Hint: Consider $g = f^{n!}$.]

- Let $g(x) = f^{n!}(x)$. Then since k divides $n!$ for each $1 \leq k \leq n$ we see that the set of periodic points of period $\leq n$ for f are all fixed points for $f^{n!}$.
- Thus, the set of fixed points of $f^{n!}$ is dense in I, so by part (a) we conclude that $f^{n!}$ is the identity function, and so every point of I is a periodic point for f.

(c) [Optional] Suppose that some iterate of f is the identity function. Show that f cannot have sensitive dependence. [Hint: If f^n is the identity, then given $x \in I$, show that there exists a positive constant ϵ_k such that $|y - x| < \epsilon_k$ implies $\left|f^k(y) - f^k(x)\right| < \beta/2$. Then let $\epsilon = \min(\epsilon_0, \epsilon_1, \dots, \epsilon_{n-1})$.]

- Suppose $x \in I$ and that f had sensitive dependence with a given value $\beta > 0$.
- Then since f (hence f^k) is continuous, we see that $\lim_{y\to x} f^k(y) = f^k(x)$.
- In particular, there is some ϵ_k such that $|y - x| < \epsilon_k$ implies $\left|f^k(y) - f^k(x)\right| < \beta/2$, by the definition of the limit.
- Now, if we let $\epsilon = \min(\epsilon_0, \epsilon_1, \dots, \epsilon_{n-1})$, then $|y - x| < \epsilon$ implies $\left|f^k(y) - f^k(x)\right| < \beta/2$ for all $0 \leq k \leq n - 1$. But since f^n is the identity, this inequality actually holds for all $k \geq 0$.
- But this contradicts sensitive dependence, because by hypothesis, for the given value of ϵ, there should exist y and k with $|y - x| < \epsilon$ such that $\left|f^k(y) - f^k(x)\right| > \beta$. (But there cannot be such y and k because $\left|f^k(y) - f^k(x)\right|$ is always less than $\beta/2$.)

(d) Suppose that f is chaotic on I. Show that f must have periodic points of arbitrarily large exact period.

- By hypothesis, f has a dense set of periodic points. If f only had periodic points of period at most n, then by part (b), some iterate of f would be the identity.
- Then by part (c), f would not have sensitive dependence. But f is chaotic, so it does have sensitive dependence. This is a contradiction, so f must have periodic points of arbitrarily large exact period.

(e) Suppose that f is chaotic on I. Show that f must have a point of exact order 2^d for every positive integer d.

- By part (d), f has points of arbitrarily large exact period. If any of these periods is not a power of 2, then by Sarkovskii's theorem we would immediately get the result, since every integer not a power of 2 precedes every power of 2 in the Sarkovskii ordering.
- If all of these periods are powers of 2, then since they become arbitrarily large, f must have a point of exact order 2^{d_i} for some sequence $d_i \to \infty$. But then, again by Sarkovskii's theorem, f also has points of order 2^k for each integer $k \leq d_i$ and each i, so since the $d_i \to \infty$ we still get the desired result.
