

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. For the following dynamical systems  $(X, f)$ , does  $f$  have sensitive dependence on all of  $X$ ? If not, does it have sensitive dependence at any individual points in  $X$ ? (Give some justification.)

- (a)  $X = [0, 1]$ ,  $f(x) = x^3$ .
  - (b)  $X = \mathbb{R}$ ,  $f(x) = 2x + 3$ .
  - (c)  $X = \mathbb{R}$ ,  $f(x) = \cos(x)$ .
  - (d)  $X = \Sigma_2$ ,  $f(d_0d_1d_2d_3d_4d_5 \dots) = (d_2d_3d_4d_5 \dots)$ .
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2. Find exact expressions for the three 4-cycles of  $q_{-2}(x) = x^2 - 2$ .
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3. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function. Answer the following:

- (a) If  $f$  has a point of exact period 11, does it necessarily have a point of exact period 32? 26? 17? 9? 4?
  - (b) If  $f$  has a point of exact period 22, does it necessarily have a point of exact period 32? 26? 17? 9? 4?
  - (c) If  $f$  has a point of exact period 16, does it necessarily have a point of exact period 32? 26? 17? 9? 4?
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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Recall the definition of the tripling map  $T(x) = \begin{cases} 3x & \text{for } 0 \leq x < 1/3 \\ 3x - 1 & \text{for } 1/3 \leq x < 2/3 \\ 3x - 2 & \text{for } 2/3 \leq x < 1 \end{cases}$  on the interval  $[0, 1)$ . Equivalently,  $T(x) = 3x$  modulo 1. The goal of this problem is to prove that  $T$  is chaotic.

- (a) Suppose  $\alpha = 0.d_1d_2d_3d_4 \dots$  in base 3, where, if  $\alpha$  has a terminating expansion, we use that expansion instead of the non-terminating one. Show that  $T(\alpha) = 0.d_2d_3d_4 \dots$ .
  - (b) Show that the periodic points for  $T$  in  $[0, 1)$  are precisely the points with a periodic base-3 decimal expansion. Conclude that the set of periodic points for  $T$  is dense in  $[0, 1)$ .
  - (c) Let  $\gamma = 0.\underbrace{012}_{\text{length 1}}\underbrace{00010210 \dots 22}_{\text{length 2}} \dots$  be the real number obtained by listing all length-1 sequences in base 3, then all length-2 sequences, then all length-3 sequences, and so forth. Show that the orbit of  $\gamma$  is dense in  $[0, 1)$ , and conclude  $T$  is transitive on  $[0, 1)$ .
  - (d) Suppose  $x$  and  $y$  are in  $[0, 1)$  with  $y > x$ . If both  $x$  and  $y$  lie in the same interval  $[0, 1/3)$ ,  $[1/3, 2/3)$ , or  $[2/3, 1)$ , show that  $|T(y) - T(x)| = 3|y - x|$ . If they lie in different intervals, show that at least one of  $|y - x|$  and  $|T(y) - T(x)|$  must be  $\geq 1/4$ .
  - (e) Suppose  $x$  and  $y$  are in  $[0, 1)$  and that  $y \neq x$ . Show that there is some  $k \geq 0$  for which  $|T^k(y) - T^k(x)| \geq 1/4$ . Conclude that  $T$  has sensitive dependence on  $[0, 1)$ .
  - (f) Show that  $T$  is chaotic on  $[0, 1)$ .
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5. Let  $g : [0, 2] \rightarrow [0, 2]$  be defined via  $g(x) = \begin{cases} x^2 - 3x + 2 & \text{for } 0 \leq x \leq 1 \\ x - 1 & \text{for } 1 < x \leq 2 \end{cases}$ . Show that  $g$  has a point of exact period  $n$  for each  $n \geq 1$ .
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6. Suppose  $f : I \rightarrow I$  is a continuous function on a closed interval  $I \subseteq \mathbb{R}$ .
- (a) Suppose that the set of fixed points of  $f$  is dense in  $I$ . Prove that  $f$  must be the identity function (i.e., that  $f(x) = x$  for all  $x \in I$ ). [Hint: If  $\lim_{n \rightarrow \infty} a_n = a$  is any convergent sequence, then  $\lim_{n \rightarrow \infty} f(a_n) = f(a)$ .]
  - (b) Suppose that the set of periodic points of period  $\leq n$  for  $f$  is dense in  $I$ . Prove that some iterate of  $f$  must be the identity function. [Hint: Consider  $g = f^{n!}$ .]
  - (c) [Optional] Suppose that some iterate of  $f$  is the identity function. Show that  $f$  cannot have sensitive dependence. [Hint: If  $f^n$  is the identity, then given  $x \in I$ , show that there exists a positive constant  $\epsilon_k$  such that  $|y - x| < \epsilon_k$  implies  $|f^k(y) - f^k(x)| < \beta/2$ . Then let  $\epsilon = \min(\epsilon_0, \epsilon_1, \dots, \epsilon_{n-1})$ .]
  - (d) Suppose that  $f$  is chaotic on  $I$ . Show that  $f$  must have periodic points of arbitrarily large exact period.
  - (e) Suppose that  $f$  is chaotic on  $I$ . Show that  $f$  must have a point of exact order  $2^d$  for every positive integer  $d$ .
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