E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 6, due Wed Feb 26th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let Σ_2 be the binary sequence space with its standard metric $d(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{\infty} \dfrac{|x_i - y_i|}{2^i}$.
   (a) Find $d(\mathbf{a},\mathbf{b})$, $d(\mathbf{a},\mathbf{c})$, and $d(\mathbf{b},\mathbf{c})$, where $\mathbf{a} = (0\overline{10})$, $\mathbf{b} = (01\overline{1})$, and $\mathbf{c} = (1\overline{10})$.
   (b) Is there a sequence $\mathbf{y}$ such that $d(\mathbf{x},\mathbf{y}) = \dfrac{1}{3}$, if $\mathbf{x} = (1\overline{0})$? Explain why or why not.
   (c) Is there a sequence $\mathbf{y}$ such that $d(\mathbf{x},\mathbf{y}) = \dfrac{1}{3}$ and $d(\mathbf{y},\mathbf{z}) = \dfrac{1}{3}$, if $\mathbf{x} = (1\overline{0})$ and $\mathbf{z} = (\overline{0})$? Explain why or why not.
   (d) Describe the points that lie in the open ball of radius 1 centered at $\mathbf{x} = (\overline{0})$: in other words, the points $\mathbf{y}$ with $d(\mathbf{x},\mathbf{y}) < 1$.

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2. Consider the shift map σ on the binary sequence space Σ_2.
   (a) How many fixed points do σ^4, σ^5, and σ^6 have?
   (b) How many cycles of length 4, 5, and 6 are there?
   (c) Find explicitly all of the 4-cycles and all of the 5-cycles.

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3. Consider the quadratic map $q_c(x) = x^2 + c$ for $c = -15/4$, let p_+ be the larger fixed point of q_c, and also let $I = [-p_+, p_+]$ and $I_0 = [-p_+, -\sqrt{-p_+ - c}]$ and $I_1 = [\sqrt{-p_+ - c}, p_+]$ be the two intervals in $q_c^{-1}(I)$. The goal of this problem is to compute some points, itinerary intervals, and cycles for the set $\Lambda = \bigcap_{n=0}^{\infty} q_c^{-n}(I)$.
   (a) Compute the intervals I, I_0, and I_1 to five decimal places.

   For a point $x \in \Lambda$, its itinerary is the sequence $(d_0 d_1 d_2 d_3 \cdots)$ where $q_c^n(x) \in I_{d_n}$ for each n.

   (b) The three points $x = -1.500000$, $x = -2.2320508$, and $x = 1.58367$ are approximations of points in Λ. Find the terms up through the digit d_{10} in the itineraries of these three points.

   For each binary string $d_0 d_1 d_2 \cdots d_n$ we now define $I_{d_0 \cdots d_n}$ to be the set of real x such that $x \in I_{d_0}$, $q_c(x) \in I_{d_1}$, $q_c^2(x) \in I_{d_2}$, ... , and $q_c^n(x) \in I_{d_n}$.

   (c) Compute the intervals I_{00}, I_{01}, I_{10}, and I_{11} to five decimal places.
   (d) Compute the intervals I_{000000}, I_{010101}, I_{101010}, and I_{110011} to five decimal places.

   We can use these interval calculations to search for periodic points: to find the point whose itinerary is a periodic cycle $(\overline{d_0 d_1 \cdots d_{n-1}})$ of length n, we can compute the interval $I_{d_0 \cdots d_{n-1}}$ and then apply Newton's method to the function $f(x) = q_c^n(x) - x$ with a starting value in that interval.

   (e) Using a starting value in I_{01}, apply Newton's method to search for a 2-cycle, and give the points on this cycle to 5 decimal places.
   (f) The points on the 2-cycle found in part (e) should lie in the intervals I_{010101} and I_{101010} computed in part (d). Verify that this is the case, and briefly explain why it is true.
   (g) To 5 decimal places, find the points on the unique 5-cycle with four points in I_0 and one point in I_1. [Hint: What is its itinerary?]

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4. Show that the given dynamical systems either are conjugate (by finding an explicit homeomorphism between them; you need not show it is actually a homeomorphism) or are not conjugate (by demonstrating a difference in their orbit structures):

   (a) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = x^3 + 6x^2 + 12x + 6$ and $g(x) = x^3$.

   (b) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = x^2$ and $g(x) = x^3$.

   (c) $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ where $f(x) = 3x$ and $g(x) = x^3$.

   (d) $(\mathbb{R}, f)$ and (Y, g) where $f(x) = 3x$, $g(x) = x^3$, and $Y = (0, \infty)$. [Hint: Exponentials.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. For the following pairs (X, S), is S a dense subset of X or not? (Give brief justification for your answers; you do not need to give a proof, just an explanation.)

   (a) $X = \mathbb{R}$, $S =$ the numbers of the form $\frac{k}{2^n}$ for k, n integers.

   (b) $X = [0, 1]$, $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.

   (c) $X = [0, 1]$, $S =$ the Cantor ternary set.

   (d) $X = [0, 1]$, $S =$ all points in X not in the Cantor ternary set.

   (e) $X = \Sigma_2$, $S =$ the sequences of the form $(d_0 d_1 d_2 \cdots)$ having only finitely many 1s.

   (f) $X = \Sigma_2$, $S =$ the orbit under σ of $\beta = (101001000100001 \cdots)$ whose expansion continues the pattern of a single 1 followed by one more 0 than the previous cluster of 0s.

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6. Let $q(x) = x^2 - 6$.

   (a) Show that q has exactly 30 real-valued 8-cycles.

   (b) Show that all of the 8-cycles lie in the interval $[-3, 3]$.

   (c) Using the function NSolve, ask Mathematica to numerically compute the roots of $q^8(x) - x = 0$. Are the results correct? Then ask Mathematica to find the roots of $\frac{q^8(x) - x}{q^4(x) - x} = 0$: are these results correct? If you have access to another computer algebra system, try asking it the same questions: does it return correct results?
