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1. If we zoom in on the orbit diagram for $q_c(x) = x^2+c$, there appears to be an attracting 3-cycle when $c = -1.76$.

   (a) Using the asymptotic orbit of the critical point $x = 0$, compute, to 10 decimal places, the apparent points on this 3-cycle.

   - Computing the 400th through 405th terms to 15 decimal places yields 0.023830803629517, −1.759432092798371, 1.335601289168856, 0.023830803629512, −1.759432092798372, 1.335601289168859.
   - These are certainly stable to 10 decimal places: we get $\boxed{\{0.0238308036, -1.7594320928, 1.3356012892\}}$.

   (b) For $f(x) = x^2 - 1.76$, verify that if $I_1 = (0.0236, 0.0241)$, $I_2 = (-1.75945, -1.75941)$, and $I_3 = (1.33552, 1.33567)$ then $f(I_1) \subseteq I_2$, $f(I_2) \subseteq I_3$, and $f(I_3) \subseteq I_1$.

   - There are no critical points of f in any of these intervals, so we need only compute the value of f on each endpoint, and verify that the endpoints of each interval land inside the next one.
   - We have $f(0.0236) = -1.75944304$ and $f(0.0239) = -1.75942879$, and these both lie in I_2.
   - Also, $f(-1.75945) = 1.3356643025$ and $f(-1.75941) = 1.3355235481$, and these both lie in I_3.
   - Finally, $f(1.33552) = 0.0236136704$ and $f(1.33567) = 0.0240143489$, and both of these lie in I_1.

   (c) Show that there is indeed an attracting 3-cycle for $q_c(x)$ when $c = -1.76$.

   - The criteria for the theorem on numerical existence of cycles are satisfied so there is a point of period 3 that lies in I_1.
   - To show it is attracting we need to compute the maximum of f' on each interval. Again, since f'' has no critical points, we only need to do these computations for the endpoints.
   - We have $f'(0.0236) = 0.0472$ and $f'(0.0239) = 0.0482$, and the maximum is 0.0482.
   - Also, $f'(-1.75945) = -3.5189$ and $f'(-1.75941) = -3.51882$, and the maximum is -3.5189.
   - Finally, $f'(1.33552) = 2.67104$ and $f'(1.33567) = 2.67134$, and the maximum is 2.67134.
   - The product of the maximum values is -0.4530886, which has absolute value less than 1. Thus, the cycle is necessarily attracting.

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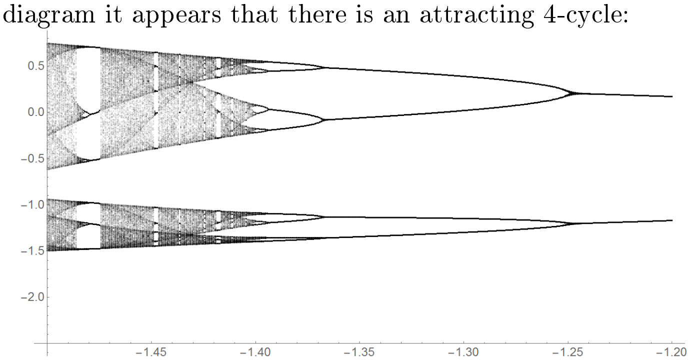
2. Consider the quadratic family $q_c(x) = x^2 + c$. For each given value of c, (i) plot the orbit diagram near that value of c, (ii) identify from the picture whether or not there seems to be an attracting cycle, (iii) numerically compute the 500th through 520th terms in the critical orbit, and (iv) if there appears to be a cycle, identify the points on it to 5 decimal places and then test whether the cycle is actually attracting.

   (a) $c = -1.34$.

   - Based on the orbit diagram it appears that there is an attracting 4-cycle:

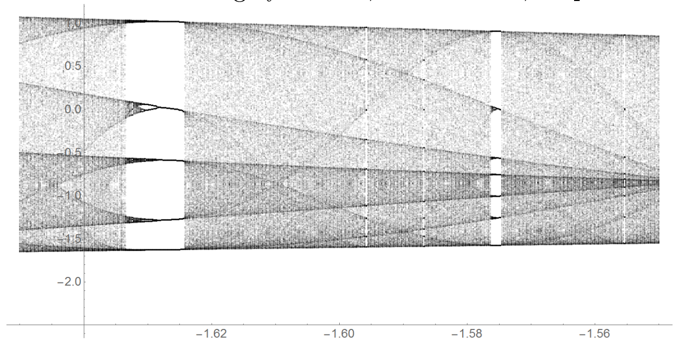


   - The terms in the critical orbit are -1.3378866, 0.4499405, -1.1375535, -0.045971937, -1.3378866, 0.4499405, -1.1375535, -0.045971937, -1.3378866, 0.4499405, -1.1375535, -0.045971937, -1.3378866, 0.4499405, -1.1375535, -0.045971937, -1.3378866, 0.4499405, -1.1375535, -0.045971937, -1.3378866.
   - Based on the orbit it seems that there is a 4-cycle whose points are -1.3378866, 0.4499405, -1.1375535, -0.045971937.

- Testing whether this cycle is attracting we compute the product of $q_c'(x)$ on the four values, which evaluates to approximately -0.5037. Since this has absolute value less than 1, the cycle is indeed attracting.

(b) $c = -1.60$.

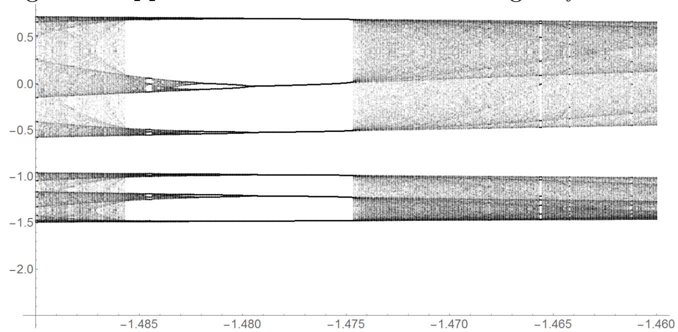
- There does not appear to be an attracting cycle here, or if there is, its period is too large to identify:



- The terms in the critical orbit are -0.49551605, -1.3544638, 0.23457231, -1.5449758, 0.78695032, -0.98070919, -0.63820949, -1.1926887, -0.17749378, -1.568496, 0.86017957, -0.86009111, -0.86024328, -0.85998151, -0.86043181, -0.85965711, -0.86098966, -0.85869681, -0.8626398, -0.85585258, -0.86751636.
- These do not appear to have any especially regular pattern, although they seem to have ventured close to the repelling fixed point $x \approx -0.86015$.

(c) $c = -1.477$.

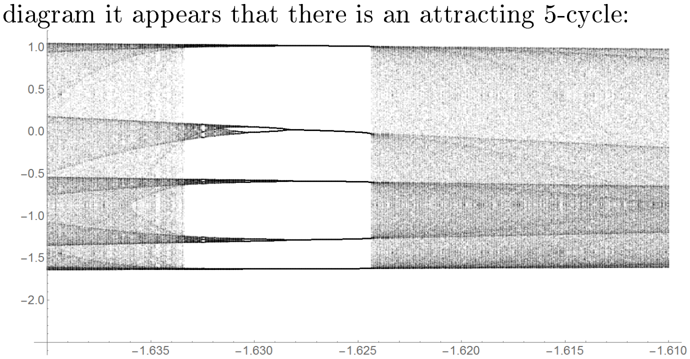
- Based on the orbit diagram it appears that there is an attracting 6-cycle.



- The terms in the critical orbit are -1.2119721, -0.0081236995, -1.476934, 0.70433406, -0.98091354, -0.51480863, -1.2119721, -0.0081236995, -1.476934, 0.70433406, -0.98091354, -0.51480863, -1.2119721, -0.0081236995, -1.476934, 0.70433406, -0.98091354, -0.51480863, -1.2119721, -0.0081236995, -1.476934.
- Based on the orbit it seems that there is a 6-cycle whose points are -1.2119721, -0.0081236995, -1.476934, 0.70433406, -0.98091354, -0.51480863.
- Testing whether this cycle is attracting, we compute the product of $q_c'(x)$ on the four values, which evaluates to approximately -0.3310. Since this has absolute value less than 1, the cycle is indeed attracting.

(d) $c = -1.626$.

- Based on the orbit diagram it appears that there is an attracting 5-cycle:



- The terms in the critical orbit are -1.2776623, 0.0064210359, -1.6259588, 1.0177419, -0.59020138, -1.2776623, 0.0064210359, -1.6259588, 1.0177419, -0.59020138, -1.2776623, 0.0064210359, -1.6259588, 1.0177419, -0.59020138, -1.2776623, 0.0064210359, -1.6259588, 1.0177419, -0.59020138, -1.2776623}
- Based on the orbit it seems that there is a 5-cycle whose points are -1.2776623, 0.0064210359, -1.6259588, 1.0177419, -0.59020138.
- Testing whether this cycle is attracting, we compute the product of $q_c'(x)$ on the four values, which evaluates to approximately -0.2564. Since this has absolute value less than 1, the cycle is indeed attracting.

Remark: For those values of c where there does appear to be a cycle, we could use a similar method as in problem 1 to prove the cycle truly exists. (But that is rather tedious and so you are not asked to do it!)

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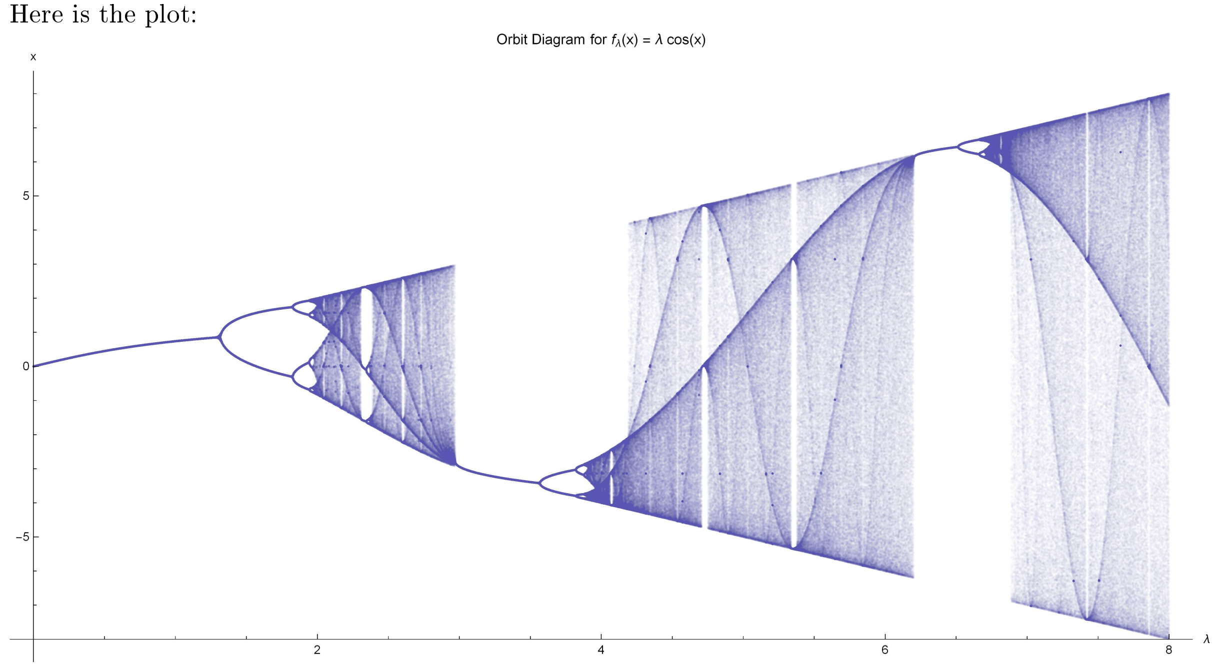
3. Consider the one-parameter family $f_\lambda(x) = \lambda\cos(x)$ for $\lambda > 0$.

   (a) Explain why there is a unique asymptotic critical orbit, and that all of its points lie in $[-\lambda, \lambda]$.

   - Since $f_\lambda'(x) = -\lambda\sin(x)$, the critical points are $x = \pi k$ for integers k.
   - Notice that $f_\lambda(\pi k) = (-1)^k\lambda$ and thus $f_\lambda^2(\pi k) = \lambda\cos(\lambda)$, meaning that each critical orbit yields the same values after two iterations: in other words, there is a unique asymptotic critical orbit.
   - All of the points of the critical orbit clearly lie in $[-\lambda, \lambda]$, since this is the range of f_λ.

   (b) Plot the orbit diagram for $0 \le \lambda \le 8$.

   - Here is the plot:



   (c) Describe the change in the orbit structure that occurs for $\lambda \approx 2.97$.

   - It appears that, after a chaotic region, an attracting fixed point appears roughly at $\lambda = 2.97$.

   (d) Describe the change in the orbit structure that occurs for$\lambda \approx 6.20$.

   - Like before, it appears that, after a chaotic region, an attracting fixed point arises at the indicated value of λ.

We now work to explain the behaviors observed in (c) and (d) and calculate more precisely where they occur.

   (e) Suppose x is an attracting fixed point of $f_\lambda(x)$. Explain why we must have $\lambda = x/\cos(x)$ and$-1 \le x\tan(x) \le 1$.

- We must have $f_\lambda(x) = x$ and also $|f_\lambda'(x)| \le 1$. The first one yields $\lambda \cos x = x$ so that $\lambda = x/\cos(x)$, and then the second one yields $|-\lambda \sin(x)| \le 1$ which after substituting $\lambda = x/\cos(x)$ yields $-1 \le x\tan(x) \le 1$, as desired.

(e) Using Newton's method (or another approach) with starting values $x = -3$ and also $x = 6.2$, estimate to 4 decimal places the values of x for which we have $-1 \le x\tan(x) \le 1$. [Hint: Find zeroes of $x\tan(x)+1$ and also of $x\tan(x) - 1$, then take the interval between them.]

- Using Newton's method with starting value $x = -3$ for the two functions $x\tan(x)+1$ and $x\tan(x)-1$, we see that $f_\lambda'(x)$ is between -1 and 1 roughly for $-3.4256 \le x \le -2.7984$.
- Using Newton's method with starting value $x = 6.2$ for the two functions $x\tan(x)+1$ and $x\tan(x)-1$, we see that $f_\lambda'(x)$ is between -1 and 1 roughly for $6.1213 \le x \le 6.4373$.

(f) Determine to 3 decimal places the range of parameter values λ near $\lambda \approx 2.97$, and also near $\lambda \approx 6.20$, for which $f_\lambda(x)$ has an attracting fixed point.

- Using the previous two parts we know we want $\lambda = x/\cos(x)$ where $-3.4256 \le x \le -2.7984$. So plotting this function on the interval we obtain $2.9717 \le \lambda \le 3.5686$. Thus, on this parameter interval, f_λ has an attracting fixed point.
- Likewise, using instead the range $6.1213 \le x \le 6.4373$, with $\lambda = x/\cos(x)$ we obtain $2.9717 \le \lambda \le 3.5686$. Thus, on this parameter interval, f_λ has an attracting fixed point.

(g) Describe the change in the orbit structure that occurs for$\lambda \approx 4.19$. Can you explain it?

- Here, it appears that the behavior is chaotic, but transitions from having the orbit be restricted to a small interval to moving through a much larger one.
- This is more difficult to prove. Ultimately, what happens is that for λ less than the transitional value, the critical orbit falls into a small interval that is mapped into itself by f_λ, whereas for λ exceeding the transitional value, the interval is no longer mapped inside itself.

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4. The goal of this problem is to investigate some properties of the Cantor ternary set $\Gamma = \bigcap_{n=0}^{\infty} C_n$, where (recall) $C_0 = [0,1]$ and C_{n+1} is obtained by removing the open middle third of each interval in C_n. Equivalently, Γ consists of the points in $[0,1]$ that have a base-3 decimal expansion containing no 1s.

(a) Find the total length of all the intervals in C_n, and show that it goes to zero exponentially fast as $n \to \infty$.

- By induction, there are 2^n intervals in C_n each of length $\frac{1}{3^n}$. So the total length is $\left(\frac{2}{3}\right)^n$, which goes to zero exponentially fast.

(b) Show that every point in Γ is equal to a limit of a sequence of other points of Γ. [Hint: Use base-3 expansions, but be careful with terminating expansions!]

- Suppose first that $x \in \Gamma$ has a non-terminating base-3 decimal expansion $0.d_1d_2d_3d_4\ldots$ where each d_i is 0 or 2.
- If we take $x_i = 0.d_1d_2\ldots d_i$, then each x_i is an element of Γ since all of its digits are 0 or 2, including the "hidden" zeroes after the terminating digit d_i.
- Clearly, $\lim_{i\to\infty} x_i = x$ by the definition of the base-3 decimal expansion, and none of the x_i is equal to x since by assumption x does not have a terminating base-3 expansion. So x is a limit of other points of Γ in this case.
- Now suppose x has a terminating expansion $x = 0.d_1d_2\cdots d_k$, and let $x_i = 0.d_1d_2\cdots d_k\underbrace{00\cdots0}_{i \text{ zeroes}}2$.

  Then clearly $\lim_{i\to\infty} x_i = x$, and each x_i is in Γ and distinct from x, so x is a limit of other points in Γ in this case as well.

(c) Show that Γ contains no nontrivial intervals (i.e., no intervals containing more than a single point). Conclude that if $x < y$ are any two points in Γ, then there exists a z with $x < z < y$ such that z is not in Γ.

- The first statement follows from the fact that the intervals at each stage of the construction of Γ have lengths shrinking to zero. Explicitly: if I is any interval of positive length ϵ, then it cannot be a subset of C_n for any n with $3^{-n} < \epsilon$, because each of the intervals in C_n has length 3^{-n}.

- The second statement is simply a reinterpretation of the first statement: if $x < y$ then because the interval $[x, y]$ is not a subset of Γ, there must be some point $z \in [x, y]$ that is missing from Γ. Since x and y are both in Γ, $z \in (x, y)$, meaning that $x < z < y$.

(d) Show that $\Gamma + \Gamma = [0, 2]$: in other words, show that every real number in the interval $[0, 2]$ can be written as the sum of two (not necessarily different) elements of Γ. [Hint: Consider the base-3 decimal expansions of elements in the set $\frac{1}{2}\Gamma = \{\frac{1}{2}x : x \in \Gamma\}$.]

- Following the hint, we notice that the elements in the set $\frac{1}{2}\Gamma$ are those real numbers that have a base-3 decimal expansion containing only 0s and 1s.
- So suppose $\beta \in [0, 2]$. Then $\alpha = \beta/2$ is in $[0, 1]$, and so has a base-3 decimal expansion of the form $0.\alpha_1\alpha_2\alpha_3\ldots$.
- Then we define x and y in $\frac{1}{2}\Gamma$ as follows: if $\alpha_i = 0$ then $x_i = y_i = 0$, if $\alpha_i = 1$ then $x_i = 1$ and $y_i = 0$, and if $\alpha_i = 2$ then $x_i = y_i = 1$.
- So, for example, if $\frac{1}{2}\alpha = 0.201202\ldots$, we would take $x = 0.101101\ldots$ and $y = 0.100101\ldots$.
- From the digit expansions we see that $\alpha = x + y$, since there are no carries and the digits all sum correctly. Then $\beta = 2x + 2y$, and clearly $2x$ and $2y$ are both elements of the Cantor ternary set Γ.

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5. Consider the "tent map" $T(x) = \begin{cases} 3x & \text{if } x \leq 1/2 \\ 3 - 3x & \text{if } 1/2 < x \end{cases}$. (Its name comes from the shape of its graph.)

(a) Show that if x is outside $[0, 1]$ then $T^n(x) \to -\infty$ as $n \to \infty$.

- If $x < 0$ then $T(x) = 3x$, so then by iterating we see $T^n(x) = 3^n x \to -\infty$.
- If $x > 1$ then $T(x) = 3 - 3x < 0$ and now we are back into the case above, so $T^n(x) \to -\infty$ also.

(b) Show that the set of points x such that $T(x) \in [0, 1]$ is the union of two closed intervals, and identify these intervals.

- Notice that T is one-to-one on $[0, 1/2]$ and $[1/2, 1]$, and T maps each of these intervals (bijectively) onto the interval $[0, 3/2]$.
- Thus, the set of points with $T(x) \in [0, 1]$ is a union of two closed intervals, one in $[0, 1/2]$ and the other in $[1/2, 1]$. Since $T(1/3) = T(2/3) = 1$ we see the intervals are $[0, 1/3]$ and $[2/3, 1]$.

(c) Show that the set of points x such that $T^2(x) \in [0, 1]$ is the union of four closed intervals, and identify these intervals.

- Since T maps each of $[0, 1/3]$ and $[2/3, 1]$ linearly onto the interval $[0, 1]$, by repeating the argument from (b) we see T^2 maps each of $[0, 1/6]$, $[1/6, 1/3]$, $[2/3, 5/6]$, and $[5/6, 1]$ linearly onto $[0, 3/2]$.
- By linearity, we see that the set of points with $T^2(x)$ is a union of the four closed intervals $[0, 1/9]$, $[2/9, 3/9]$, $[6/9, 7/9]$, and $[8/9, 1]$.

(d) Identify the set of points x such that $T^n(x) \in [0, 1]$ for every $n \geq 1$. [Optional: Prove it.]

- Based on the pattern from parts (b) and (c) we guess that the desired set of points is the Cantor ternary set Γ.
- To show this, we will prove by induction on n that the set of points such that $T^n(x) \in [0, 1]$ is the set C_n from the construction of the Cantor set. We also include as part of our induction hypothesis the statement that T^n maps each of the 2^n intervals in C_n bijectively onto the interval $[0, 1]$.
- We already did the base case $n = 1$ in part (b). For the inductive step, suppose that C_n is the set of points such that $T^n(x) \in [0, 1]$ and that T^n maps each of the 2^n intervals in C_n bijectively onto the interval $[0, 1]$.
- Then the set of points mapped into $[0, 1]$ by T^{n+1} consists of the points in C_n, but without the points x such that $T^n(x) \in (1/3, 2/3)$. By linearity and the induction hypothesis, this latter set consists of the set obtained by removing the open middle third of each of the intervals in C_n: in other words, the set C_{n+1}. Also, by linearity, each of the resulting 2^{n+1} intervals is mapped bijectively onto $[0, 1]$ by T^{n+1}. This is what was we needed to show for the induction, so we are done.
