E. Dummit's Math 3543 ∼ Dynamics, Chaos, and Fractals, Spring 2025 ∼ Homework 5, due Wed Feb 19th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. If we zoom in on the orbit diagram for $q_c(x) = x^2+c$, there appears to be an attracting 3-cycle when $c = -1.76$.

   (a) Using the asymptotic orbit of the critical point $x = 0$, compute, to 10 decimal places, the apparent points on this 3-cycle.

   (b) For $f(x) = x^2 - 1.76$, verify that if $I_1 = (0.0236, 0.0241)$, $I_2 = (-1.75945, -1.75941)$, and $I_3 = (1.33552, 1.33567)$ then $f(I_1) \subseteq I_2$, $f(I_2) \subseteq I_3$, and $f(I_3) \subseteq I_1$.

   (c) Show that there is indeed an attracting 3-cycle for $q_c(x)$ when $c = -1.76$.

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2. Consider the quadratic family $q_c(x) = x^2 + c$. For each given value of c, (i) plot the orbit diagram near that value of c, (ii) identify from the picture whether or not there seems to be an attracting cycle, (iii) numerically compute the 500th through 520th terms in the critical orbit, and (iv) if there appears to be a cycle, identify the points on it to 5 decimal places and then test whether the cycle is actually attracting.

   (a) $c = -1.34$.

   (b) $c = -1.60$.

   (c) $c = -1.477$.

   (d) $c = -1.626$.

   Remark: For those values of c where there does appear to be a cycle, we could use a similar method as in problem 1 to prove the cycle truly exists. (But that is rather tedious and so you are not asked to do it!)

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3. Consider the one-parameter family $f_\lambda(x) = \lambda \cos(x)$ for $\lambda > 0$.

   (a) Explain why there is a unique asymptotic critical orbit, and that all of its points lie in $[-\lambda, \lambda]$.

   (b) Plot the orbit diagram for $0 \le \lambda \le 8$.

   (c) Describe the change in the orbit structure that occurs for $\lambda \approx 2.97$.

   (d) Describe the change in the orbit structure that occurs for$\lambda \approx 6.20$.

   We now work to explain the behaviors observed in (c) and (d) and calculate more precisely where they occur.

   (e) Suppose x is an attracting fixed point of $f_\lambda(x)$. Explain why we must have $\lambda = x/\cos(x)$ and$-1 \le x\tan(x) \le 1$.

   (e) Using Newton's method (or another approach) with starting values $x = -3$ and also $x = 6.2$, estimate to 4 decimal places the values of x for which we have $-1 \le x\tan(x) \le 1$. [Hint: Find zeroes of $x\tan(x) + 1$ and also of $x\tan(x) - 1$, then take the interval between them.]

   (f) Determine to 3 decimal places the range of parameter values λ near $\lambda \approx 2.97$, and also near $\lambda \approx 6.20$, for which $f_\lambda(x)$ has an attracting fixed point.

   (g) Describe the change in the orbit structure that occurs for$\lambda \approx 4.19$. Can you explain it?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. The goal of this problem is to investigate some properties of the Cantor ternary set $\Gamma = \bigcap_{n=0}^{\infty} C_n$, where (recall) $C_0 = [0, 1]$ and C_{n+1} is obtained by removing the open middle third of each interval in C_n. Equivalently, Γ consists of the points in $[0, 1]$ that have a base-3 decimal expansion containing no 1s.

   (a) Find the total length of all the intervals in C_n, and show that it goes to zero exponentially fast as $n \to \infty$.

   (b) Show that every point in Γ is equal to a limit of a sequence of other points of Γ. [Hint: Use base-3 expansions, but be careful with terminating expansions!]

   (c) Show that Γ contains no nontrivial intervals (i.e., no intervals containing more than a single point). Conclude that if $x < y$ are any two points in Γ, then there exists a z with $x < z < y$ such that z is not in Γ.

   (d) Show that $\Gamma + \Gamma = [0, 2]$: in other words, show that every real number in the interval $[0, 2]$ can be written as the sum of two (not necessarily different) elements of Γ. [Hint: Consider the base-3 decimal expansions of elements in the set $\frac{1}{2}\Gamma = \{\frac{1}{2}x : x \in \Gamma\}$.]

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5. Consider the "tent map" $T(x) = \begin{cases} 3x & \text{if } x \leq 1/2 \\ 3 - 3x & \text{if } 1/2 < x \end{cases}$. (Its name comes from the shape of its graph.)

   (a) Show that if x is outside $[0, 1]$ then $T^n(x) \to -\infty$ as $n \to \infty$.

   (b) Show that the set of points x such that $T(x) \in [0, 1]$ is the union of two closed intervals, and identify these intervals.

   (c) Show that the set of points x such that $T^2(x) \in [0, 1]$ is the union of four closed intervals, and identify these intervals.

   (d) Identify the set of points x such that $T^n(x) \in [0, 1]$ for every $n \geq 1$. [Optional: Prove it.]
