E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 4, due Wed Feb 5th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each function f and fixed point x_0,
(i) verify that the given point x_0 is a fixed point that is attracting or weakly attracting,
(ii) find the immediate attracting basin I of x_0 to three decimal places, and then
(iii) plot the sets $f^{-1}(I)$, $f^{-2}(I)$, $f^{-3}(I)$, $f^{-4}(I)$, $f^{-5}(I)$ to give an approximation of the full attracting basin.

   (a) $f(x) = 0.5x - 2x^2 + x^3$, with fixed point $x_0 = 0$.
   (b) $f(x) = 0.5x - 2.5x^2 + x^3$, with fixed point $x_0 = 0$.
   (c) $f(x) = -0.5x + 2.5x^2 - x^3$, with fixed point $x_0 = 1.5$.

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2. Each of the given one-parameter families $f_\lambda(x)$ has a bifurcation at the given value of λ_0. Plot the bifurcation diagram for the family and use it to identify the type of bifurcation there, and then show algebraically that the claimed bifurcation does occur (make sure to include all relevant calculations for this verification!):

   (a) $f_\lambda(x) = \lambda \sin(x)$, $\lambda_0 = -1$.
   (b) $f_\lambda(x) = \lambda e^x - 2$, $\lambda_0 = e$.
   (c) $f_\lambda(x) = e^{\lambda x}$, $\lambda_0 = -e$. [Hint: Use the algebraic equations characterizing the bifurcation to find its exact coordinates.]
   (d) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 2$.
   (e) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 4/\sqrt{3}$.

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3. Compute the Schwarzian derivative $Sf(x)$ for each function $f(x)$, and decide whether $Sf(x) < 0$ for all x:

   (a) $f_1(x) = \dfrac{x}{x+1}$.
   (b) $f_2(x) = x^a$ for a a constant. (The answer will depend on a.)
   (c) $f_3(x) = \tan(x)$.
   (d) $f_4(x) = x^4 - 3x + \pi$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let $f_\lambda(x) = \lambda x - x^3$.

   (a) Plot the bifurcation diagram for this family. (Include fixed points and 2-cycles.)
   (b) Identify the three pairs (λ_0, x_0) where period-doubling bifurcations occur, and then show algebraically that period-doubling bifurcations do occur there.
   (c) There is another bifurcation at $\lambda_0 = 1$ that is called a "pitchfork" bifurcation. Explain why it is not a saddle-node bifurcation, according to our definition.

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5. The goal of this problem is to illustrate how the Schwarzian derivative was originally used in complex analysis to characterize the fractional linear transformations $f(x) = \frac{ax+b}{cx+d}$, where a, b, c, d are constants.

   (a) Show that $S(1/x) = 0$ and $S(cx+d) = 0$ for any c, d.

   (b) Suppose that $Sf = 0$ and $Sg = 0$. Show that $S(f \circ g) = 0$ also. [Hint: Schwarzian chain rule.]

   (c) Show that $S(\frac{1}{cx+d}) = 0$ and then that $S(\frac{ax+b}{cx+d}) = S(\frac{(bc-ad)/c}{cx+d} + \frac{a}{c}) = 0$ for any a, b, c, d. [Hint: Use (a) and (b).]

   Part (c) shows that the Schwarzian derivative of any fractional linear transformation is zero. Our goal now is to show the converse: that a function with Schwarzian derivative zero must be a fractional linear transformation.

   (d) [Optional] Suppose f is such that $Sf(x) = 0$ for all x. Show that $\ln(f'') = \frac{3}{2}\ln(f') + C$ for some constant C. [Hint: Integrate $\frac{f'''}{f''} = \frac{3}{2}\frac{f''}{f'}$.]

   (e) [Optional] Suppose g is such that $\ln(g') = \frac{3}{2}\ln(g) + C$ for some constant C. Show that $g(x) = (cx+d)^{-2}$ for some c and d. [Hint: Integrate $g^{-3/2}g' = e^C$.]

   (f) [Optional] Suppose $Sf(x) = 0$ for all x. Show that $f(x) = \frac{ax+b}{cx+d}$ for some a, b, c, d.
