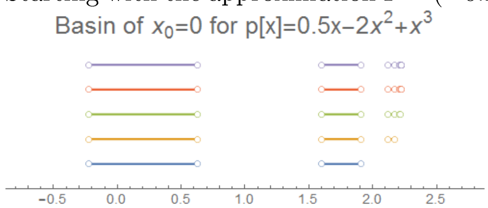
1. For each function f and fixed point x_0,
   (i) verify that the given point x_0 is a fixed point that is attracting or weakly attracting,
   (ii) find the immediate attracting basin I of x_0 to three decimal places, and then
   (iii) plot the sets $f^{-1}(I)$, $f^{-2}(I)$, $f^{-3}(I)$, $f^{-4}(I)$, $f^{-5}(I)$ to give an approximation of the full attracting basin.

   (a) $f(x) = 0.5x - 2x^2 + x^3$, with fixed point $x_0 = 0$.

      i. We see $f(0) = 0$ so x_0 is a fixed point, and $f'(0) = 0.5$ so it is attracting.

      ii. Note f is defined everywhere, and solving $f(x) = x$ yields the two other real fixed points $x = -0.22474$ and $x = 2.22474$. The real preimages of the first fixed point (in addition to itself) are $x = 0.62516$ and $x = 1.59958$ while the second fixed point only has itself as a preimage. Finally, there are no real 2-cycles. Since the nearest points on either side of 0 are -0.22474 and 0.62516, the immediate basin is $\boxed{(-0.224, 0.625)}$.

      iii. Starting with the approximation $I = (-0.224, 0.625)$, the desired inverse images are as plotted below.

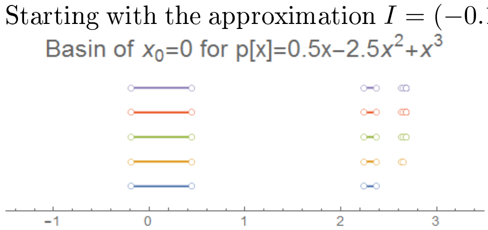


   (b) $f(x) = 0.5x - 2.5x^2 + x^3$, with fixed point $x_0 = 0$.

      i. We see $f(0) = 0$ so x_0 is a fixed point, and $f'(0) = -0.5$ so it is attracting.

      ii. Note f is defined everywhere, and solving $f(x) = x$ yields the two other real fixed points $x = -0.1861$ and $x = 2.6861$. The real preimages of the first fixed point (in addition to itself) are $x = 0.4465$ and $x = 2.2396$ while the second fixed point has no other real preimages. Finally, there are no real 2-cycles. Since the nearest points on either side of 0 are -0.1861 and 0.4465, the immediate basin is $\boxed{(-0.1861, 0.4465)}$.

      iii. Starting with the approximation $I = (-0.186, 0.446)$, the desired inverse images are as plotted below.

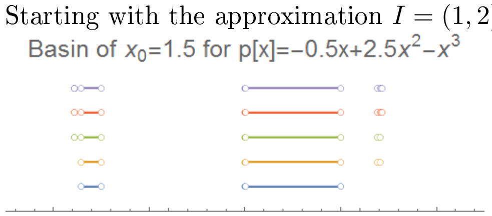


   (c) $f(x) = -0.5x + 2.5x^2 - x^3$, with fixed point $x_0 = 1.5$.

      i. We see $f(1.5) = 1.5$ so x_0 is a fixed point, and $f'(1.5) = 0.25$ so it is attracting.

      ii. Note f is defined everywhere, and solving $f(x) = x$ yields the two other real fixed points $x = 0$ and $x = 1$. The real preimages of $x = 0$ in addition to itself are $x = 0.21922$ and $x = 2.28078$ while the real preimages of $x = 1$ in addition to itself are $x = -0.5$ and $x = 2$. Finally, there is a single real 2-cycle $\{-0.78671, 2.42753\}$. Since the nearest points on either side of 1.5 are 1 and 2, the immediate basin is $\boxed{(1, 2)}$.

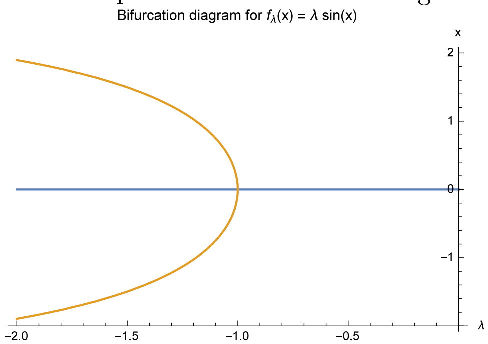
      iii. Starting with the approximation $I = (1, 2)$, the desired inverse images are as plotted below.



2. Each of the given one-parameter families $f_\lambda(x)$ has a bifurcation at the given value of λ_0. Plot the bifurcation diagram for the family and use it to identify the type of bifurcation there, and then show algebraically that the claimed bifurcation does occur (make sure to include all relevant calculations for this verification!):

   (a) $f_\lambda(x) = \lambda \sin(x)$, $\lambda_0 = -1$.

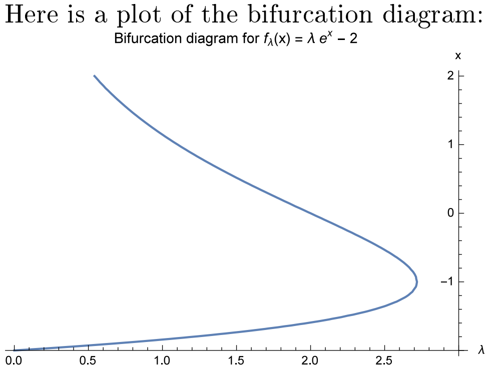
   - Here is a plot of the bifurcation diagram:



   - In the graph, the fixed-point curve "sprouts" a curve of period-2 points when $\lambda_0 = -1$ and $x_0 = 0$, which indicates a period-doubling bifurcation.
   - To show it algebraically, we need to check that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = -1$, and $\left.\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
   - We have $f_{-1}(0) = 0$, $f'_{-1}(0) = -1$, and $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda}(0) = 2\lambda$ which is nonzero at $\lambda_0 = -1$.
   - All of the criteria are satisfied, so there is a $\boxed{\text{period-doubling bifurcation}}$ at $\lambda_0 = -1$.

   (b) $f_\lambda(x) = \lambda e^x - 2$, $\lambda_0 = e$.

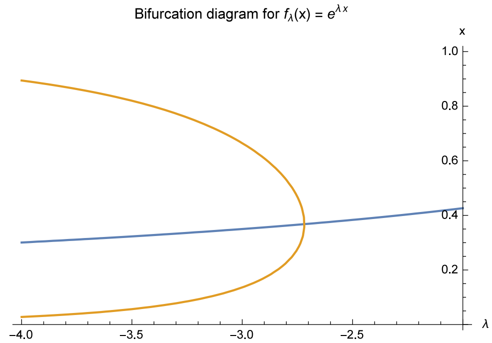
   - Here is a plot of the bifurcation diagram:



   - In the graph, a pair of fixed points seem to show up $\lambda_0 = e$ and $x_0 = -1$, which indicates a saddle-node bifurcation. (There do not appear to be any period-2 points at all.)
   - To show it algebraically, we need to check that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$, and $\left.\dfrac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
   - We have $f_e(-1) = -1$, $f'_e(-1) = 1$, $f''_e(-1) = -1$, and $\dfrac{\partial f_\lambda}{\partial \lambda} = e^x$ which is nonzero everywhere.
   - All of the criteria are satisfied, so there is a $\boxed{\text{saddle-node bifurcation}}$ at $\lambda_0 = e$.

   (c) $f_\lambda(x) = e^{\lambda x}$, $\lambda_0 = -e$. [Hint: Use the algebraic equations characterizing the bifurcation to find its exact coordinates.]

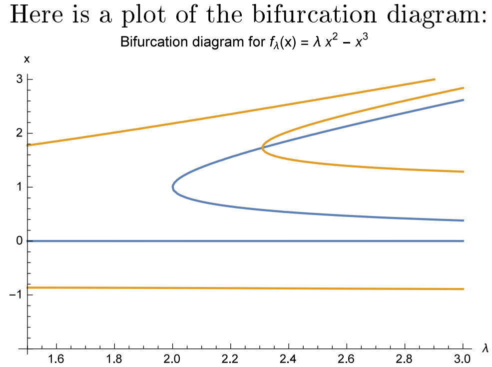
   - Here is a plot of the bifurcation diagram:



- In the graph, the fixed-point curve "sprouts" a curve of period-2 points when $\lambda_0 = -e$, which indicates a period-doubling bifurcation. It is not so easy to see the exact value of x_0 from the picture (although it appears to be approximately 0.38).
- Instead, we solve for it algebraically: we want x_0 for which $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) = -1$, or equivalently, $e^{-ex} = x$ and $-e \cdot e^{-ex} = -1$.
- Plugging the first equation into the second gives $-ex = -1$ so that $x = 1/e$, which does indeed work in both equations. (Since $1/e$ is about 0.368, it seems right.)
- We can compute $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda} = \lambda e^{\lambda e^{\lambda x} + \lambda x}\left(\lambda^2 x e^{\lambda x} + \lambda e^{\lambda x} + \lambda x + 2\right)$, so the value at (λ_0, x_0) simplifies to $-1/e$, which is nonzero.
- All of the criteria are satisfied, so there is a $\boxed{\text{period-doubling bifurcation}}$ at $\lambda_0 = -e$.

(d) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 2$.

- Here is a plot of the bifurcation diagram:



- In the graph, a pair of fixed points seem to show up at $\lambda_0 = 2$ and $x_0 = 1$, which indicates a saddle-node bifurcation.
- To show it algebraically, we need to check that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$, and $\left.\dfrac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
- We have $f_2(1) = 1$, $f'_2(1) = 1$, $f''_2(1) = -2$, and $\dfrac{\partial f_\lambda}{\partial \lambda} = x^2$ which is nonzero for $x_0 = 1$.
- All of the criteria are satisfied, so there is a $\boxed{\text{saddle-node bifurcation}}$ at $\lambda_0 = 2$.

(e) $f_\lambda(x) = \lambda x^2 - x^3$, $\lambda_0 = 4/\sqrt{3}$.

- From the diagram in part (d), since $4/\sqrt{3}$ is approximately 2.31, there appears to be a period-doubling bifurcation. We cannot see the exact value from the picture, so we will solve for it algebraically.
- We want x_0 for which $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) = -1$, or equivalently, $\frac{4}{\sqrt{3}}x^2 - x^3 = x$ and $\frac{8}{\sqrt{3}}x - 3x^2 = -1$. Solving with a computer shows that the only common root is $x = \sqrt{3}$.
- We can compute $\dfrac{\partial (f_\lambda^2)'}{\partial \lambda} = 2x^3\left(6\lambda^2 - 12x^4 + 21\lambda x^3 + \left(3 - 9\lambda^2\right)x^2 - 10\lambda x\right)$, and setting $x_0 = \sqrt{3}$ and $\lambda_0 = 4/\sqrt{3}$ gives the value as $6\sqrt{3}$. (This is best done by computer, obviously!)
- All of the criteria are satisfied, so there is a $\boxed{\text{period-doubling bifurcation}}$ at $\lambda_0 = 4/\sqrt{3}$.

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3. Compute the Schwarzian derivative $Sf(x)$ for each function $f(x)$, and decide whether $Sf(x) < 0$ for all x:

(a) $f_1(x) = \dfrac{x}{x+1}$.

- We have $f'(x) = (x+1)^{-2}$, $f''(x) = -2(x+1)^{-3}$, $f'''(x) = 6(x+1)^{-4}$ so $\dfrac{f'''}{f'} - \dfrac{3}{2}\left(\dfrac{f''}{f'}\right)^2 = 6(x+1)^{-2} - \dfrac{3}{2}\cdot(-2)^2(x+1)^{-2} = \boxed{0}$.
- So it is not true that $Sf(x) < 0$ for all x. (Or any x for that matter.)

(b) $f_2(x) = x^a$ for a a constant. (The answer will depend on a.)

- We have $f' = a\,x^{a-1}$, $f'' = a(a-1)x^{a-2}$, $f''' = a(a-1)(a-2)x^{a-3}$ so $Sf = \dfrac{f'''}{f'} - \dfrac{3}{2}\left(\dfrac{f''}{f'}\right)^2 = (a-1)(a-2)x^{-2} - \dfrac{3}{2}(a-1)^2x^{-2} = \boxed{\dfrac{1-a^2}{2}x^{-2}}$.
- So for $|a| > 1$ it is true that $Sf(x) < 0$ for all x, and for $|a| \le 1$ it is not true that $Sf(x) < 0$ for all x.

(c) $f_3(x) = \tan(x)$.

- We have $f' = \sec(x)^2$, $f'' = 2\sec(x)^2\tan(x)$, $f''' = 2\sec(x)^4 + 4\sec(x)^2\tan(x)^2$ so $Sf = \dfrac{f'''}{f'} - \dfrac{3}{2}\left(\dfrac{f''}{f'}\right)^2 = 2\sec(x)^2 + 4\tan(x)^2 - \dfrac{3}{2}(2\tan(x))^2 = 2\sec(x)^2 - 2\tan(x)^2 = \boxed{2}$.
- So it is not true that $Sf(x) < 0$ for all x. (Or any x for that matter.)

(d) $f_4(x) = x^4 - 3x + \pi$.

- We have $f' = 4x^3 - 3$, $f'' = 12x^2$, $f''' = 24x$ so $Sf = \dfrac{f'''}{f'} - \dfrac{3}{2}\left(\dfrac{f''}{f'}\right)^2 = \dfrac{24x}{4x^3-3} - \dfrac{3}{2}\cdot\dfrac{144x^4}{(4x^3-3)^2} = \boxed{-\dfrac{24x(3+5x^3)}{(4x^3-3)^2}}$.
- So it is not true that $Sf < 0$ for all x, since for instance if $x = -0.1$ then the value is positive.

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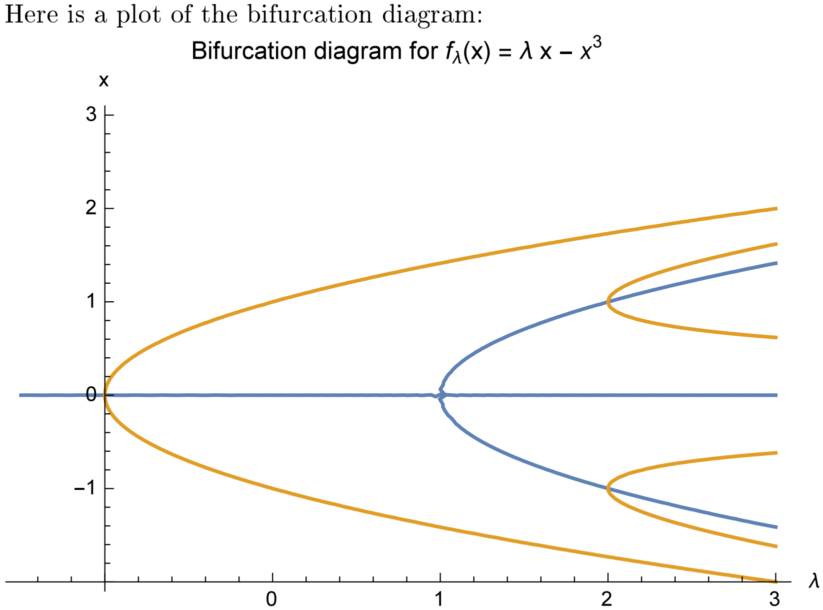
4. Let $f_\lambda(x) = \lambda x - x^3$.

(a) Plot the bifurcation diagram for this family. (Include fixed points and 2-cycles.)

- Here is a plot of the bifurcation diagram:



(b) Identify the three pairs (λ_0, x_0) where period-doubling bifurcations occur, and then show algebraically that period-doubling bifurcations do occur there.

- In fact there are three period-doubling bifurcations: one is at $(\lambda_0, x_0) = (-1, 0)$ and the other two are at $(\lambda_0, x_0) = (2, \pm 1)$.
- To apply the criterion, we check $f_{\lambda_0}(x_0) = x_0$, that $f'_{\lambda_0}(x_0) = -1$, and that $\left.\frac{\partial (f_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$.
- We have $f_{-1}(0) = 0$ and $f'_{-1}(0) = -1$, so these clearly hold. Furthermore, it is straightforward to compute that $(f_\lambda^2)'(0) = \lambda^2$, so the desired derivative is $\frac{\partial (f_\lambda^2)'}{\partial \lambda}(0) = 2\lambda$, which is indeed nonzero at $\lambda_0 = -1$.
- Similarly, $f_2(\pm 1) = \pm 1$ and $f'_2(\pm 1) = -1$, so since $(f_\lambda^2)'(0) = \lambda^2$ the criteria also hold at the other two points.

(c) There is another bifurcation at $\lambda_0 = 1$ that is called a "pitchfork" bifurcation. Explain why it is not a saddle-node bifurcation, according to our definition.

- The problem is that, in addition to the two fixed points that are created, there is also the constant-valued fixed point at $x = 0$.
- Our definition of a saddle-node bifurcation requires that there be no fixed points on one side of the bifurcation, one point at the bifurcation, and two points on the other side. Instead, we have one fixed point on the left and three on the right.
- Note that the saddle-node criterion is *not* an if-and-only-if statement: saddle-node bifurcations can still occur even when the criterion fails (which it does in this case: we have $f_\lambda(x) = x$ and $f'_\lambda(x) = 1$ but $f''_\lambda(x) = 0$).

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5. The goal of this problem is to illustrate how the Schwarzian derivative was originally used in complex analysis to characterize the fractional linear transformations $f(x) = \frac{ax+b}{cx+d}$, where a, b, c, d are constants.

(a) Show that $S(1/x) = 0$ and $S(cx + d) = 0$ for any c, d.

- For $f = \frac{1}{x}$ we have $f' = -\frac{1}{x^2}$, $f'' = \frac{2}{x^3}$, $f''' = -\frac{6}{x^4}$ so $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \frac{6}{x^2} - \frac{3}{2}(\frac{2}{x})^2 = 0$.
- Likewise, for $f = cx + d$ we have $f' = c$ and $f'' = f''' = 0$ so $Sf = \frac{0}{c} - \frac{3}{2}(\frac{0}{c})^2 = 0$.

(b) Suppose that $Sf = 0$ and $Sg = 0$. Show that $S(f \circ g) = 0$ also. [Hint: Schwarzian chain rule.]

- By the Schwarzian chain rule we know if $h = f \circ g$, then $Sh(x) = Sf(g(x)) \cdot g'(x)^2 + Sg(x)$.
- Thus, if $Sf = 0$ and $Sg = 0$, then $S(f \circ g) = Sf(g(x)) \cdot g'(x)^2 + Sg(x) = 0 \cdot g'(x)^2 + 0 = 0$ as well.

(c) Show that $S(\frac{1}{cx+d}) = 0$ and then that $S(\frac{ax+b}{cx+d}) = S(\frac{(bc-ad)/c}{cx+d} + \frac{a}{c}) = 0$ for any a, b, c, d. [Hint: Use (a) and (b).]

- Applying (b) with $f = 1/x$ and $g = cx + d$ yields $S(\frac{1}{cx+d}) = 0$ by (a).
- Then applying (b) with $f = \frac{1}{cx+d}$ and $g = \frac{bc-ad}{c}x + \frac{a}{c}$ yields $S(\frac{ax+b}{cx+d}) = S(\frac{(bc-ad)/c}{cx+d} + \frac{a}{c}) = 0$, as desired.
- Of course, we could just have done the algebra directly: the purpose was to see how the Schwarzian chain rule can save on effort.

Part (c) shows that the Schwarzian derivative of any fractional linear transformation is zero. Our goal now is to show the converse: that a function with Schwarzian derivative zero must be a fractional linear transformation.

(d) [Optional] Suppose f is such that $Sf(x) = 0$ for all x. Show that $\ln(f'') = \frac{3}{2}\ln(f') + C$ for some constant C. [Hint: Integrate $\frac{f'''}{f''} = \frac{3}{2}\frac{f''}{f'}$.]

- If $Sf = 0$ then from the definition we see $\frac{f'''}{f'} = \frac{3}{2}\left(\frac{f''}{f'}\right)^2$, which is equivalent to $\frac{f'''}{f''} = \frac{3}{2}\frac{f''}{f'}$.
- Integrating both sides yields $\ln(f'') = \frac{3}{2}\ln(f') + C$ for some constant of integration C.

(e) [Optional] Suppose g is such that $\ln(g') = \frac{3}{2}\ln(g) + C$ for some constant C. Show that $g(x) = (cx+d)^{-2}$ for some c and d. [Hint: Integrate $g^{-3/2}g' = e^C$.]

- The given equation is equivalent to $\ln(g') = \ln(g^{3/2} \cdot e^C)$, which upon exponentiation yields $g' = g^{3/2} \cdot e^C$.
- If we rewrite this expression as $g^{-3/2}g' = e^C$ and integrate both sides, we obtain $-\frac{1}{2}g^{-1/2} = e^C x + D$ for some constants C and D.
- Setting $c = -2e^C$ and $d = -2D$ then produces $g^{-1/2} = cx + d$, so that $g(x) = (cx+d)^{-2}$.

(f) [Optional] Suppose $Sf(x) = 0$ for all x. Show that $f(x) = \frac{ax+b}{cx+d}$ for some a, b, c, d.

- By part (a) we see that $\ln(f'') = \frac{3}{2}\ln(f') + C$. By part (b) with $g = f'$, we see that $f'(x) = (cx+d)^{-2}$ for some c and d.
- Integrating (yet again) gives $f(x) = \frac{-1/c}{cx+d} + A$ for some constant A. This is of the form $\frac{ax+b}{cx+d}$, as required.
