1. For each function, (i) verify that the given point x_0 is a fixed point that is attracting or weakly attracting, (ii) compute exactly or to at least 5 decimal places all of the other real fixed points and points where the function is undefined, the preimages of those other fixed/undefined points, and the 2-cycles, (iii) find the immediate attracting basin of x_0, and (iv) find the largest interval around x_0 for which $\left|\frac{f(x)-x_0}{x-x_0}\right| < 1$ for all $x \neq x_0$ in that interval and verify that this interval is contained in the immediate attracting basin.

   (a) $f(x) = \frac{3}{4}x + x^3$, with fixed point $x_0 = 0$.

   i. We see $f(0) = 0$ so x_0 is a fixed point, and $f'(0) = 3/4$ so it is attracting.

   ii. Note f is defined everywhere, and solving $f(x) = x$ yields the other fixed points $x = \boxed{-\frac{1}{2}, \frac{1}{2}}$. Each of these points has only itself as a real preimage (this can be checked numerically, or it follows because f is strictly increasing, so $f(x) = r$ has only one real solution for any r). Finally, solving $f(f(x)) = x$ yields only the fixed points so there are no 2-cycles.

   iii. By (ii), the list of possible endpoints for the immediate basin is only $\{-\infty, -1/2, 1/2, \infty\}$, so taking the ones closest to x_0 on either side, we see that the immediate basin is $\boxed{(-1/2, 1/2)}$.

   iv. Solving $\left|\frac{f(x)-0}{x-0}\right| < 1$ yields $\left|x^2 + \frac{3}{4}\right| < 1$ so that $-1 < x^2 + \frac{3}{4} < 1$ so that $x^2 < \frac{1}{4}$ yielding $-\frac{1}{2} < x < \frac{1}{2}$. This is the interval $\boxed{(-1/2, 1/2)}$ which is in fact the entire immediate attracting basin!

   (b) $f(x) = x - x^5$, with fixed point $x_0 = 0$.

   i. We see $f(0) = 0$ so x_0 is a fixed point, and $f'(0) = 1$ so it is neutral. Since the first nonzero derivative of f is the 5th derivative and $f^{(5)}(0) = -120$, this neutral fixed point is weakly attracting on both sides.

   ii. Note f is defined everywhere, and solving $f(x) = x$ yields only the given fixed point $x = 0$. Finally, solving $f(f(x)) = x$ yields the real fixed point $x = 0$ along with the 2-cycle $\{-2^{1/4}, 2^{1/4}\}$.

   iii. By (i), the list of possible endpoints for the immediate basin is only $\{-\infty, -2^{1/4}, 2^{1/4}, \infty\}$, so taking the ones closest to x_0 on either side, we see that the immediate basin is $\boxed{(-2^{1/4}, 2^{1/4})}$.

   iv. Solving $\left|\frac{f(x)-0}{x-0}\right| < 1$ yields we get $\left|1 - x^4\right| < 1$ so that $0 \leq x^4 < 2$, which is equivalent to $x \in (-2^{1/4}, 2^{1/4})$. So the desired interval is $\boxed{(-2^{1/4}, 2^{1/4})}$ which is in fact the entire immediate attracting basin!

   (c) $f(x) = \frac{2x^2}{3x-1}$, with fixed point $x_0 = 0$.

   i. Note f is undefined only at $x = \boxed{1/3}$. Solving $f(x) = x$ yields the other fixed point $x = \boxed{1}$. Then solving $f(x) = 1$ yields the preimage points $x = \boxed{1/2}$ and $x = 1$, and solving $f(x) = 1/3$ yields no real solutions. Finally, solving $f(f(x)) = x$ yields only the real fixed points $x = 0, 1$ so there are no 2-cycles.

   ii. We see $f(0) = 0$ so x_0 is a fixed point, and $f'(0) = 0$ so it is (super)attracting.

   iii. By (i), the list of possible endpoints for the immediate basin is only $\{-\infty, 1/3, 1/2, 1, \infty\}$, so taking the ones closest to x_0 on either side, we see that the immediate basin is $\boxed{(-\infty, 1/3)}$.

   iv. Solving $\left|\frac{f(x)-0}{x-0}\right| < 1$ yields $\left|\frac{2x}{3x-1}\right| < 1$ so that $-1 < \frac{2x}{3x-1} < 1$ yielding $x < \frac{1}{5}$ or $x > 1$. So the desired interval around 0 is $\boxed{(-\infty, 1/5)}$ which is contained in the immediate basin, but not actually equal to it.

(d) $f(x) = \cos x$, with fixed point $x_0 \approx 0.739085$ (to six decimal places). [Skip item (iv) for this part.]

i. Note f is defined everywhere. Solving $f(x) = x$ numerically yields only the fixed point $x_0 \approx 0.739085$. Finally, solving $f(f(x)) = x$ numerically yields only the fixed point x_0 again, so there are no 2-cycles.

ii. We see $f(x_0) = x_0$ so x_0 is a fixed point, and $f'(x_0) = -\sin(x_0) \approx -0.673612$ so it is attracting.

iii. By (i), the list of possible endpoints for the immediate basin is only $\{-\infty, \infty\}$, so the immediate basin is $\boxed{(-\infty, \infty)}$.

Remark: In fact the inequality $\left|\frac{f(x)-x_0}{x-x_0}\right| < 1$ is in fact true for every real number $x \neq x_0$ as can be seen by graphing the function $\frac{f(x)-x_0}{x-x_0}$, or as follows by an application of the mean value theorem: we have $\frac{f(x)-x_0}{x-x_0} = f'(c)$ for some c between x_0 and x, but $f'(c) = -\sin c$ will have absolute value at most 1 since it is the sine of a real number.

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2. For each function f and each initial value x_0, apply Newton's method starting at x_0 to search for a zero of the function f, giving the results of the 10th, 100th, and 101st iterations to 5 decimal places. Does it appear that Newton's method has identified a zero of the function?

(a) $f(x) = \cos x$ with $x_0 = 0.1$.

- We have $N(x) = x - \frac{p(x)}{p'(x)} = x - \frac{\cos x}{-\sin x} = x + \cot x$.
- Iterating $N(x)$ with starting value $x_0 = 0.1$, we see that to 5 decimal places, the 10th, 100th, and 101st iterations are all $\boxed{10.99557}$. It seems that Newton's method $\boxed{\text{has identified a zero}}$ in this case (in this case, $7\pi/2$).

(b) $f(x) = e^x - 20.25x$ with $x_0 = 0.1$.

- We have $N(x) = x - \frac{p(x)}{p'(x)} = x - \frac{e^x - 20.25x}{e^x - 20.25}$.
- Iterating $N(x)$ with starting value $x_0 = 0.1$, we see that to 5 decimal places, the 10th, 100th, and 101st iterations are all $\boxed{0.05202}$. It seems that Newton's method $\boxed{\text{has identified a zero}}$ in this case.

(c) $f(x) = e^x - 20.25x$ with $x_0 = 5.1$.

- We have $N(x) = x - \frac{e^x - 20.25x}{e^x - 20.25}$.
- Iterating $N(x)$ with starting value $x_0 = 5.1$, we see that to 5 decimal places, the 10th, 100th, and 101st iterations are all $\boxed{4.51572}$. It seems that Newton's method $\boxed{\text{has identified a zero}}$ in this case.

(d) $f(x) = x^2 + 1$ with $x_0 = 0.1$.

- We have $N(x) = x - \frac{x^2+1}{2x}$.
- Iterating $N(x)$ with starting value $x_0 = 0.1$, we see that to 5 decimal places, the 10th iteration is $\boxed{-0.04108}$, the 100th iteration is $\boxed{154.83139}$, and the 101st iteration is $\boxed{77.41247}$. It seems that the method $\boxed{\text{has not found a zero}}$ in this case. (That should not be surprising, since clearly f has no real zeroes!)

(e) $f(x) = x^3 - 3x^2 + 3x - 1$ with $x_0 = 0.1$.

- We have $N(x) = x - \frac{x^3 - 3x^2 + 3x - 1}{3x^2 - 6x + 3}$.
- Iterating $N(x)$ with starting value $x_0 = 0.1$, we see that to 5 decimal places, the 10th iteration is $\boxed{0.98439}$ and the 100th and 101st iterations are $\boxed{1.00000}$. It seems that the method $\boxed{\text{has found a zero}}$ in this case, but it has taken longer than in the previous cases: that is because $f(x) = (x-1)^3$, so the actual zero is $x = 1$ of multiplicity 3.

(f) $f(x) = x^3 - 2x + 2$ with $x_0 = 0.1$.

- We have $N(x) = x - \frac{x^3 - 2x + 2}{3x^2 - 2}$.
- Iterating $N(x)$ with starting value $x_0 = 0.1$, we see that to 5 decimal places, the 10th iteration is $\boxed{0.00020}$, the 100th iteration is $\boxed{0.00000}$, and the 101st iteration is $\boxed{1.00000}$. It seems that the method $\boxed{\text{has not found a zero}}$ in this case, but instead gotten caught in a 2-cycle.

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3. Let $p(x) = x^5(x-1)^2(x-2)$.

(a) Find the Newton iterating function $N(x)$ for p, along with the fixed points of N.

- We have $N(x) = x - \dfrac{p(x)}{p'(x)} = x - \dfrac{x(x-1)(x-2)}{5(x-1)(x-2) + 2x(x-2) + x(x-1)} = \boxed{x - \dfrac{x(x-1)(x-2)}{8x^2 - 20x + 10}}$ after some cancellation.
- From the expression (or the arguments we gave about multiplicities of roots in Newton's fixed point theorem) the fixed points of N are simply the roots $x = \boxed{0, 1, 2}$ of f.

(b) Give a table of the first 10 iterates of the Newton iterating function applied to the starting values 0.1, 1.1, and 2.1. Which root has the fastest convergence? Which root has the slowest convergence? What does Newton's fixed point theorem predict?

- Here is such a table:

0.1000000000	1.1000000000	2.1000000000
0.0788366337	1.0573275862	2.0295731707
0.0623704717	1.0313908199	2.0034092406
0.0494700769	1.0165822158	2.0000513882
0.0393130935	1.0085509080	2.0000000118
0.0312869102	1.0043464409	2.0000000000
0.0249271528	1.0021918285	2.0000000000
0.0198773374	1.0011006818	2.0000000000
0.0158612274	1.0005515477	2.0000000000
0.0126632525	1.0002760775	2.0000000000
0.0101142766	1.0001381149	2.0000000000

- Clearly, 0 is the slowest, 1 is in the middle, and 2 is the fastest. Newton's fixed point theorem says that roots with higher multiplicity will have slower convergence (since $N'(r) = 1 - \dfrac{1}{k}$ where k is the multiplicity of the root), and that is what we see.

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4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and that x is a (weakly) attracting fixed point of f.

(a) Show that the immediate attracting basin of x cannot have another attracting fixed point y as one of its endpoints. [Hint: Consider what the immediate attracting basin of y would be.]

- Suppose otherwise, so that the immediate attracting basin of x has one of its endpoints at an attracting fixed point y. But then the immediate attracting basin of y is an open interval containing y, so it would intersect with the attracting basin of x: but then orbits starting in that overlap would have to attract to both x and y, impossible.

(b) Show that the immediate attracting basin of x has one of the following forms: (i) $(-\infty, \infty)$, (ii) $(-\infty, a)$ or (a, ∞) where a is a repelling or neutral fixed point of f, (iii) (a, b) where a and b are repelling or neutral fixed points of f, (iv) (a, b) where one of a, b is a repelling or neutral fixed point of f and f maps the other to it, or (v) (a, b) where $\{a, b\}$ is a 2-cycle of f.

- As noted in (a), the finite endpoints of the immediate attracting basin cannot be another attracting fixed point. So the possibilities are (i) the interval is unbounded on both ends hence is $(-\infty, \infty)$, (ii) it is bounded at one end (where that endpoint must be a fixed point that is not attracting) hence is of the form $(-\infty, a)$ or (a, ∞) where a is a repelling or neutral fixed point of f, or both endpoints are finite.
- If both endpoints are finite, either (iii) they are both non-attracting fixed points, or (iv) one is a non-attracting fixed point and the other is a preimage of it, or (v) they form a 2-cycle. These are all of the possibilities, so we are done.

Remark: The result of (b) shortens the list of possibilities that need to be considered when computing immediate basins using the methods that we discussed in class.

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5. Let $f(x) = x^3 - 2x - 2$.

(a) Show that this polynomial has exactly one real root r and that it lies in the interval $(1, 2)$.

- Since $f(1) = -3$ and $f(2) = 2$ the intermediate value theorem says r has a root in $(1, 2)$.
- Since $f'(x) = 3x^2 - 2$, f has a local maximum at $-\sqrt{2/3}$ and a local minimum at $\sqrt{2/3}$. Since $f(-\sqrt{2/3}) = \sqrt{\dfrac{32}{27}} - 2$ is negative, f has no real root on $(-\infty, \sqrt{2/3})$. Since it is monotone increasing on $(\sqrt{2/3}, \infty)$, it has exactly one root.

(b) Find the Newton iterating function for f and then use Newton's method to calculate the root r, accurate to 10 decimal places.

- We have $N(x) = x - \dfrac{f(x)}{f'(x)} = \boxed{x - \dfrac{x^3 - 2x - 2}{3x^2 - 2} = \dfrac{2(x^3 + 1)}{3x^2 - 2}}$.
- The orbit of 1 is 1, 4, 2.826086956521739, 2.146719013739235, 1.842326277140092, 1.772847636439237, 1.769301397436449, 1.769292354297359, 1.769292354238631, 1.769292354238631, ...
- So to 10 decimal places the value is $\boxed{1.7692923542}$.

(c) Compare your result in part (b) to the numerical value of $\dfrac{1}{3}\left[\sqrt[3]{27 + 3\sqrt{57}} + \sqrt[3]{27 - 3\sqrt{57}}\right]$.

- To 10 decimal places, the quantity is 1.7692923542. (This seems vaguely familiar....)

Remark: In fact we can show that this value is actually a root of f: write $a = \sqrt[3]{27 + 3\sqrt{57}}$ and $b = \sqrt[3]{27 - 3\sqrt{57}}$. Then $ab = \sqrt[3]{27^2 - 3^2 \cdot 57} = \sqrt[3]{216} = 6$ and $a^3 + b^3 = (27 + 3\sqrt{57}) + (27 - 3\sqrt{57}) = 54$. Expanding the cube gives $(a + b)^3 = a^3 + b^3 + 3ab(a + b) = 54 + 18(a + b)$. So if $3y = (a + b)$, then $(3y)^3 = 54 + 18(3y)$, or $27y^3 = 54 + 54y$, or $y^3 - 2y - 2 = 0$. In other words: $y = \dfrac{1}{3}(a + b)$ is a root of the cubic.

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6. Suppose f is continuously differentiable and has finitely many zeroes $r_1 < r_2 < \cdots < r_n$ each having finite multiplicity ≥ 1.

(a) Show that the Newton iterating function $N(x)$ is undefined somewhere in the interval (r_i, r_{i+1}) for each i with $1 \leq i \leq n - 1$. [Hint: Use the mean value theorem.]

- By the mean value theorem, since $f(r_i) = 0 = f(r_{i+1})$, there is a $c \in (r_i, r_{i+1})$ for which $f'(c) = 0$.
- Furthermore, c cannot be a zero of f since it is not one of the r_i. Then $N(c) = c - \dfrac{f(c)}{f'(c)}$, and this quantity is undefined because $f(c) \neq 0$ but $f'(c) = 0$.

(b) If $i \neq 1, n$, show that the immediate attracting basin for r_i as a fixed point of N must have the form (a, b) where $\{a, b\}$ is a 2-cycle for N. [Hint: Explain why the immediate basin does not extend to $\pm\infty$, and then use this to show that N cannot be undefined at either endpoint of the basin.]

- By assumption, all of the zeroes of f have finite multiplicity ≥ 1, so by Newton's fixed point theorem they are all attracting. Thus, none of them can be the endpoint of any immediate basin, since otherwise we would have an overlap between those immediate basins.
- Since $i \neq 1$, the immediate basin of r_i cannot extend to $-\infty$ (otherwise it would include r_1, which is nonsense) and similarly since $i \neq n$, the immediate basin of r_i cannot extend to $-\infty$ (otherwise it would include r_n). So it is an interval (a, b).
- If N were undefined at a, then we would necessarily have $f'(a) = 0$. Since $f(a) \neq 0$ because a is not a fixed point of N (i.e., a zero of f), we see that $\lim_{x \to a+} N(x)$ is either ∞ or $-\infty$. But now since (a, r_i) is part of the immediate basin, every point in $f((a, r_i))$ has orbit attracted to r_i, so the immediate basin of r_i would be infinite. Since we know this is not the case, N must be defined at a, and similarly N must also be defined at b.
- So the only possibility for (a, b) is to have $\{a, b\}$ form a 2-cycle, as claimed.

(c) For $f(x) = x(x-1)(x-4)$, find the immediate attracting basin for the fixed point $r_2 = 1$ of the Newton iterating function for f. (Give your answer to five decimal places.)

- The function satisfies the hypotheses of part (b) so all we need to do is compute the 2-cycles of $N(x) = \dfrac{2x^3 - 5x^2}{3x^2 - 10x + 4}$.
- We can do this numerically (using Newton's method, even, in an amusing meta-application of the procedure) to see that there is a single real-valued 2-cycle $\{0.532104, 2.362762\}$.
- Thus, the immediate basin is $\boxed{(0.53211, 2.36276)}$, where we rounded "inwards".
