E. Dummit's Math 3543 ∼ Dynamics, Chaos, and Fractals, Spring 2025 ∼ Homework 3, due Wed Jan 29th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each function f and given point x_0,
   (i) verify that the given point x_0 is a fixed point that is attracting or weakly attracting,
   (ii) compute exactly or to at least 5 decimal places all of the other real fixed points and points where the function is undefined, the preimages of those other fixed/undefined points, and the 2-cycles,
   (iii) find the immediate attracting basin of x_0, and
   (iv) find the largest interval around x_0 for which $\left|\dfrac{f(x)-x_0}{x-x_0}\right| < 1$ for all $x \neq x_0$ in that interval and verify that this interval is contained in the immediate attracting basin.

   (a) $f(x) = \dfrac{3}{4}x + x^3$, with fixed point $x_0 = 0$.
   (b) $f(x) = x - x^5$, with fixed point $x_0 = 0$.
   (c) $f(x) = \dfrac{2x^2}{3x-1}$, with fixed point $x_0 = 0$.
   (d) $f(x) = \cos x$, with fixed point $x_0 \approx 0.739085$ (to six decimal places). [Skip item (iv) for this part.]

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2. For each function f and each initial value x_0, apply Newton's method starting at x_0 to search for a zero of the function f, giving the results of the 10th, 100th, and 101st iterations to 5 decimal places. Does it appear that Newton's method has identified a zero of the function?

   (a) $f(x) = \cos x$ with $x_0 = 0.1$.
   (b) $f(x) = e^x - 20.25x$ with $x_0 = 0.1$.
   (c) $f(x) = e^x - 20.25x$ with $x_0 = 5.1$.
   (d) $f(x) = x^2 + 1$ with $x_0 = 0.1$.
   (e) $f(x) = x^3 - 3x^2 + 3x - 1$ with $x_0 = 0.1$.
   (f) $f(x) = x^3 - 2x + 2$ with $x_0 = 0.1$.

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3. Let $p(x) = x^5(x-1)^2(x-2)$.

   (a) Find the Newton iterating function $N(x)$ for p, along with the fixed points of N.
   (b) Give a table of the first 10 iterates of the Newton iterating function applied to the starting values 0.1, 1.1, and 2.1. Which root has the fastest convergence? Which root has the slowest convergence? What does Newton's fixed point theorem predict?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and that x is a (weakly) attracting fixed point of f.

   (a) Show that the immediate attracting basin of x cannot have another attracting fixed point y as one of its endpoints. [Hint: Consider what the immediate attracting basin of y would be.]

   (b) Show that the immediate attracting basin of x has one of the following forms: (i) $(-\infty, \infty)$, (ii) $(-\infty, a)$ or (a, ∞) where a is a repelling or neutral fixed point of f, (iii) (a, b) where a and b are repelling or neutral fixed points of f, (iv) (a, b) where one of a, b is a repelling or neutral fixed point of f and f maps the other to it, or (v) (a, b) where $\{a, b\}$ is a 2-cycle of f.

   Remark: The result of (b) shortens the list of possibilities that need to be considered when computing immediate basins using the methods that we discussed in class.

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5. Let $f(x) = x^3 - 2x - 2$.

   (a) Show that this polynomial has exactly one real root r and that it lies in the interval $(1, 2)$.

   (b) Find the Newton iterating function for f and then use Newton's method to calculate the root r, accurate to 10 decimal places.

   (c) Compare your result in part (b) to the numerical value of $\frac{1}{3}\left[\sqrt[3]{27+3\sqrt{57}} + \sqrt[3]{27-3\sqrt{57}}\right]$.

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6. Suppose f is continuously differentiable and has finitely many zeroes $r_1 < r_2 < \cdots < r_n$ each having finite multiplicity ≥ 1.

   (a) Show that the Newton iterating function $N(x)$ is undefined somewhere in the interval (r_i, r_{i+1}) for each i with $1 \leq i \leq n-1$. [Hint: Use the mean value theorem.]

   (b) If $i \neq 1, n$, show that the immediate attracting basin for r_i as a fixed point of N must have the form (a, b) where $\{a, b\}$ is a 2-cycle for N. [Hint: Explain why the immediate basin does not extend to $\pm\infty$, and then use this to show that N cannot be undefined at either endpoint of the basin.]

   (c) For $f(x) = x(x-1)(x-4)$, find the immediate attracting basin for the fixed point $r_2 = 1$ of the Newton iterating function for f. (Give your answer to five decimal places.)
