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1. Each of these functions has a neutral fixed point. Find it (you do not need to show it is unique), and then determine whether it is weakly attracting or weakly repelling for orbits on each side, making sure to include brief justification for the behavior:

   (a) $m(x) = x + x^5$.

   - Clearly 0 is the only fixed point, and it is neutral since $m'(0) = 1$. Since $m^{(5)}(0) = 120$ is the first nonzero higher derivative, we have $k = 5$, so the neutral fixed point theorem says that $\boxed{\text{0 is weakly repelling}}$.

   (b) $a(x) = x - x^4 + x^7$.

   - The fixed points are the solutions to $x = x - x^4 + x^7$ so that $x^4(x^3 - 1) = 0$. So $x = 0$ and $x = 1$ are the possibilities, but since $a'(0) = 1$ and $a'(1) = 3$ only $\boxed{\text{0 is neutral}}$.
   - Since $a'(0) = 1$, $a''(0) = a'''(0) = 0$, and $a^{(4)}(0) = -24$, we have $k = 4$, so the neutral fixed point theorem says that 0 is $\boxed{\text{weakly attracting on the right}}$ and $\boxed{\text{weakly repelling on the left}}$.

   (c) $t(x) = \sin(x)$.

   - The fixed points are the solutions to $x = \sin(x)$. Clearly $x = 0$ is a solution, and there are no other solutions because the function $x - \sin(x)$ is monotone increasing (its derivative is $1 - \cos(x)$ which is always nonnegative). And indeed, $t'(0) = 1$, so $\boxed{\text{0 is neutral}}$.
   - We have $t''(0) = 0$ and $t'''(0) = -1$, so $k = 3$ and then the neutral fixed point theorem says that 0 is $\boxed{\text{weakly attracting}}$.

   (d) $h(x) = e^{x/e}$.

   - From a plot (or a rootfinder, or inspired guessing) we see that $x = e$ is a fixed point of h, and there are no other fixed points because the function $e^{x/e} - x$ has a global minimum at $x = e$ (its derivative is $e^{(x/e)-1} - 1$ which is positive for $x > e$ and negative for $x < e$). Indeed, $h'(e) = 1$, so $\boxed{e \text{ is neutral}}$.
   - We have $h''(e) = 1/e$ so we get $k = 2$ and then the neutral fixed point theorem says that e is $\boxed{\text{weakly repelling on the right}}$ and $\boxed{\text{weakly attracting on the left}}$.

   (e) $y(x) = \ln(1 - x)$.

   - From a plot we see that $x = 0$ is a fixed point of y, and there are no other solutions because the function $x - \ln(1 - x)$ is monotone increasing (its derivative is $1 + \dfrac{1}{1-x}$ which is positive on the whole domain). Then $y'(0) = -1$ so $\boxed{\text{0 is neutral}}$.
   - Since $y'(0) = -1$ we need to look at $q(x) = y^2(x) = \ln(1 - \ln(1 - x))$. Some calculus gives $q'(0) = 1$, $q''(0) = 0$, and $q'''(0) = 1$, so $k = 3$ and the neutral fixed point theorem says that 0 is weakly repelling for q and hence also $\boxed{\text{weakly repelling}}$ for y.

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2. Let $p_c(x) = -x + x^2 + cx^3$, where c is a constant, and notice that 0 is a neutral fixed point. Determine (in terms of c) when 0 is weakly attracting and when it is weakly repelling.

   - Since $p_c'(0) = -1$ we need to look at $q(x) = p_c^2(x) = x - (2c + 2)x^3 + (c + 1)x^4 + \cdots + c^4x^9$.
   - If $2c + 2 > 0$ then the coefficient of x^3 is negative and so the neutral fixed point theorem says that 0 is weakly attracting.
   - If $2c + 2 < 0$ then the coefficient of x^3 is positive and so the neutral fixed point theorem says that 0 is weakly repelling.
   - If $c = -1$ then we compute explicitly $p_{-1}^2(x) = x + 4x^5 - 6x^6 + 6x^7 - 3x^8 + x^9$, so the neutral fixed point theorem says that 0 is weakly repelling.
   - Thus, $\boxed{\text{0 is weakly attracting for } c > -1}$ and $\boxed{\text{0 is weakly repelling for } c \le -1}$.

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3. Let $p(x) = x^3 - ax$ for a parameter $a > 1$.

(a) Find the three fixed points of p and classify them as attracting, repelling, or neutral (in terms of a).

- The fixed points are $x = 0$ and $x = \pm\sqrt{a+1}$. We have $p'(0) = -a$ so since $a > 1$ this point is $\boxed{\text{repelling}}$. Also, $p'(\pm\sqrt{a+1}) = 3 + 2a$, so these points are also $\boxed{\text{repelling}}$.

(b) Find a 2-cycle for p of the form $\{x_0, -x_0\}$ and classify it as attracting, repelling, or neutral (in terms of a).

- Solving $x^3 - ax = -x$ gives $x = 0$ and $x = \pm\sqrt{a-1}$, and so $\boxed{\{\sqrt{a-1}, -\sqrt{a-1}\}}$ is a 2-cycle.
- Also, $p'(\pm\sqrt{a-1}) = 2a - 3$, so we need to compare $\left|p'(\sqrt{a-1})p'(-\sqrt{a-1})\right| = (2a-3)^2$ to 1. It is less than 1 when $1 < a < 2$, equal to 1 when $a = 2$, and greater than 1 when $a > 2$.
- Thus, the 2-cycle is $\boxed{\text{attracting when } 1 < a < 2}$, $\boxed{\text{neutral when } a = 2}$, and $\boxed{\text{repelling when } a > 2}$.

(c) For $a = 5/2$, it turns out that p has two other 2-cycles in addition to the one you found in part (b). Compute them explicitly, and classify them as attracting, repelling, or neutral.

- At first glance, $p(p(x)) - x$ is a polynomial of degree 9. However, we know it is divisible by the cubic $p(x) - x$, and it is also divisible by the quadratic polynomial arising from the 2-cycle we found in part (b), namely $x^2 - (a-1)$.
- Thus, the quotient is a polynomial of degree 4. After some algebra (ideally, with a computer), we obtain $\dfrac{p(p(x)) - x}{[p(x) - x] \cdot [x^2 - (a-1)]} = x^4 - ax^2 + 1$.
- This quartic polynomial is easy to solve since it is quadratic in x^2: the solutions are $x = \pm\sqrt{\dfrac{a \pm \sqrt{a^2 - 4}}{2}}$ for the four possible choices of signs. (Note that there will only be real-valued solutions when $a \geq 2$.)
- When $a = 5/2$, we get $x = \pm\sqrt{2}$, $\pm 1/\sqrt{2}$, so the two other 2-cycles are $\boxed{\{\sqrt{2}, -1/\sqrt{2}\} \text{ and } \{-\sqrt{2}, 1/\sqrt{2}\}}$.
- We have $p'(\pm\sqrt{2}) = \dfrac{7}{2}$ and $p'(\pm 1/\sqrt{2}) = -1$ so $\boxed{\text{both 2-cycles are repelling}}$.

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4. Suppose $\{x_0, x_1, x_2\}$ is a neutral 3-cycle for the function f and $f'(x_i) = 1$ for each $i = 0, 1, 2$.

(a) If $g = f^3$, show that $g''(x_0) = g''(x_1) = g''(x_2) = f''(x_0) + f''(x_1) + f''(x_2)$. [Hint: Use the product and chain rules to compute $g''(x)$, then set $x = x_0$.]

- We compute $g'(x) = f'(f^2(x)) \cdot f'(f(x)) \cdot f'(x)$, and then $g''(x) = f''(f^2(x)) \cdot f'(f(x))^2 \cdot f'(x)^2 + f'(f^2(x)) \cdot f''(f(x)) \cdot f'(x)^2 + f'(f^2(x)) + f'(f(x)) \cdot f'(f(x)) \cdot f''(x)$.
- Setting $x = x_0$ produces $g''(x_0) = f''(x_2) + f''(x_1) + f''(x_0)$, since all of the f' terms are equal to 1 by hypothesis. Similarly, $g''(x_1)$ and $g''(x_2)$ are also equal to this quantity, by symmetry.

(b) Let $p(x) = 1 + x - 3x^2 - \dfrac{15}{4}x^3 + \dfrac{3}{2}x^4 + \dfrac{9}{4}x^5$. Show that 0 lies on a neutral 3-cycle for p, and classify the behavior of p^3 near 0 as weakly attracting or repelling on each side of 0. [Hint: Use (a).]

- We have $p(0) = 1$, $p(1) = -1$, and $p(-1) = 0$, so $\{0, 1, -1\}$ is a 3-cycle. Also, $p'(x) = 1 - 6x - \dfrac{45}{4}x^2 + 6x^3 + \dfrac{45}{4}x^4$, so $p'(0) = 1$, $p'(1) = 1$, and $p'(-1) = 1$, so the cycle is neutral.
- For the classification, we need to compute the second derivative of $g = p^3$. We can use the result of part (a) since all the hypotheses hold.
- Since $p''(x) = -6 - \dfrac{45}{2}x + 18x^2 + 45x^3$, we get $g''(0) = p''(0) + p''(1) + p''(-1) = (-6) + (69/2) + (-21/2) = 18$. Since this is positive, by the neutral fixed point theorem we see that the 3-cycle is $\boxed{\text{weakly attracting on the left of 0}}$ and $\boxed{\text{weakly repelling on the right of 0}}$.
- It is worth noting that if we had tried to use the neutral point theorem directly, we would have needed to compute the second derivative of the polynomial $p^3(x)$, which has degree 125. (Not pleasant!)

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5. Consider the two functions

$$\begin{aligned} s_1(x) &= -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{653}{1000}x^{10} \\ s_2(x) &= -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{652}{1000}x^{10} \end{aligned}$$

and observe that they both have a neutral fixed point at $x_0 = 0$.

(a) Numerically compute the first five elements in the orbits of 0.1 and 0.01 under both s_1 and s_2 to at least 30 decimal places.

- Here are the results of the computations taken to 40 decimal places. We have grouped the even and odd terms together in the orbits to make it easier to compare them.

Orbits under s_1	Even terms (x_0, x_2, x_4)	Odd terms (x_1, x_3, x_5)
0.1	0.1000000000000000000000000000000000000000	-0.0908726423098000000000000000000000000000
	0.1000000000085596382404114674105599083957	-0.0908726423168626335514823106525346698502
	0.1000000000171192764899588293710982820028	-0.0908726423239252671103901149797925650902
0.01	0.0100000000000000000000000000000000000000	-0.0099009852476251737434800000000000000000
	0.0100000000000000000000020676287351274485	-0.0099009852476251737434820268842987805353
	0.0100000000000000000000041352574702548970	-0.0099009852476251737434840537685975610706

Orbits under s_2	Even terms (x_0, x_2, x_4)	Odd terms (x_1, x_3, x_5)
0.1	0.1000000000000000000000000000000000000000	-0.0908726423097000000000000000000000000000
	0.1000000000084768421444246387415192088427	-0.0908726423166943177570824311499090033597
	0.1000000000169536842978217079382026613107	-0.0908726423236886355214576483591304298254
0.01	0.0100000000000000000000000000000000000000	-0.0099009852476251737434700000000000000000
	0.0100000000000000000000009194338836903823	-0.0099009852476251737434709013156331975496
	0.0100000000000000000000018388677673807646	-0.0099009852476251737434718026312663950992

(b) Based only on the orbits of 0.1 and 0.01, decide whether you think 0 is weakly attracting or weakly repelling for s_1 and s_2.

- For s_1 and s_2, both orbits are moving away from 0, so it appears that 0 weakly repelling for both functions.

(c) Now compute the five elements in the orbit of 0.001 under s_1 and s_2 to 40 decimal places. Do the results agree with your guess from part (b)?

- Here are the results of the computations taken to 40 decimal places, grouped the same way as before:

Orbit under s_1	Even terms (x_0, x_2, x_4)	Odd terms (x_1, x_3, x_5)
0.001	0.0010000000000000000000000000000000000000	-0.0009990009985024976250017374993480000000
	0.0010000000000000000000000000000000073619	-0.0009990009985024976250017374993480073471
	0.0010000000000000000000000000000000147237	-0.0009990009985024976250017374993480146943

Orbit under s_2	Even terms (x_0, x_2, x_4)	Odd terms (x_1, x_3, x_5)
0.001	0.0010000000000000000000000000000000000000	-0.0009990009985024976250017374993470000000
	0.0009999999999999999999999999999999954156	-0.0009990009985024976250017374993469954248
	0.0009999999999999999999999999999999908312	-0.0009990009985024976250017374993469908496

- For s_1, the orbits are still moving away from 0, but for s_2, the orbits now seem to be moving towards 0, though we need almost all of the 40 digits' worth of accuracy to see that this is happening. It seems that 0 might actually be weakly attracting for s_2 rather than weakly repelling as we previously thought.

(d) Show that the neutral fixed point is weakly repelling for one of s_1, s_2 but weakly attracting for the other.

- Using Mathematica to compute the Taylor series for the double iterates to degree 11 shows that $s_1^2(x) = x + \frac{29}{4000}x^{11} + O(x^{12})$, while $s_2^2(x) = x - \frac{19}{4000}x^{11} + O(x^{12})$, where the $O(x^{12})$ is standard notation for "terms of degree at least 12 in x".
- Thus by the neutral fixed point criterion, 0 is weakly repelling for s_1 but weakly attracting for s_2.

- It should be a little bit unsettling that the computations for s_2 lie to us unless we take 40 digits' worth of accuracy! It should also be a bit unsettling that a change of 0.001 in the coefficient of x^{10} in the polynomial $s_1(x)$ is able to change the behavior from weakly repelling to weakly attracting.

Remark: In fact, it is completely impossible to analyze the behavior of s_1 and s_2 near 0 using a graph.

- On the interval $[-\epsilon, \epsilon]$ for small ϵ, the maximum value of the difference $|s_1(s_1(x)) - x|$ is roughly on the order of $11! \cdot \epsilon^{11}$ by Taylor's theorem.
- So, for example, if $\epsilon = 10^{-3}$, the maximum value of the difference between $s_1(s_1(x))$ and x will be roughly of size $11! \cdot 10^{-33} \approx 10^{-25}$.
- Even if the computer draws 10^5 points in the interval when producing the graph (a typical HD video has pixel dimensions 1920×1080, about $1/50$th as many points), the difference between $y = s_1(s_1(x))$ and $y = x$ will be about 20 orders of magnitude smaller than the points the computer can draw! (Even if we throw in an enormous fudge factor of 10^{10}, there is still no way to see the difference.)

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6. The goal of this problem is to explore a pathological example: a fixed point of a non-differentiable function. Let $f(x) = \begin{cases} x + x\sin\left(\frac{1}{x}\right) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$: note that f maps $[0, \infty) \to [0, \infty)$, and that f is continuous but not differentiable at $x = 0$.

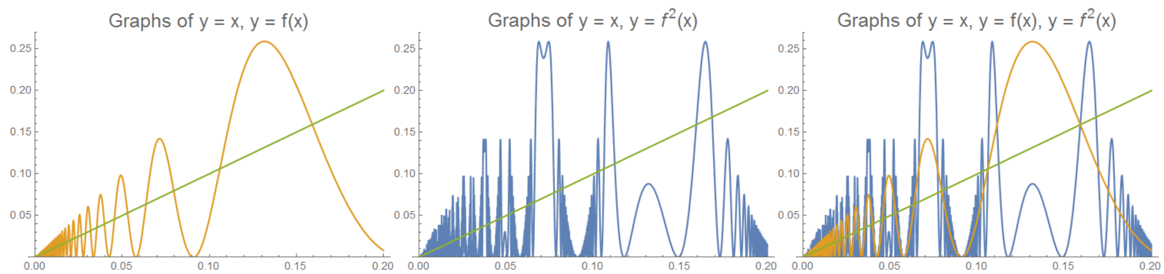
   (a) Find all the fixed points of f, and show that (except for $x = 0$) they are all repelling.

   - Clearly $x = 0$ is a fixed point. Setting $x = x + x\sin\left(\frac{1}{x}\right)$ with $x > 0$ gives $x\sin\left(\frac{1}{x}\right) = 0$, which is true only when $\sin\left(\frac{1}{x}\right) = 0$ – namely, for $x = \boxed{\frac{1}{k\pi} \text{ for } k \text{ a positive integer}}$.
   - We have $f'(x) = 1 + \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$, so $f'(\frac{1}{k\pi}) = 1 - k\pi \cdot (-1)^k$, which always has absolute value larger than 1 for each positive integer k: thus, they are all repelling.

   (b) Numerically compute the first 10 elements in the orbits of each of $x_0 = 0.1$, 0.01, 0.001, 0.0001, and 0.00015494157427205179. Are they being attracted to or repelled from zero in a consistent way?

   - Mathematica gives the following results (rounded to six significant digits but with 80-digit internal precision):
     - {0.100000, 0.0455979, 0.0483462, 0.0950198, 0.0103655, 0.0185837, 0.0112875, 0.0179274, 0.00547075, 0.00845900, 0.000692953}
     - {0.0100000, 0.00493634, 0.00986559, 0.0171554, 0.0340603, 0.00393475, 0.00518508, 0.000308758, 0.000370067, 0.000528406, 0.00102881}
     - {0.00100000, 0.00182688, 0.00306457, 0.00182712, 0.00296465, 0.000250055, 0.000283750, 0.000115410, 0.000145071, 0.000218322, 0.000208997}
     - {0.000100000, 0.0000694386, 0.0000804566, 0.000144181, 0.0000300359, 0.00000382582, 0.00000729502, 0.00000438300, 0.00000153916, 0.00000205480, 0.00000359648}
     - {0.000154942, 0.000300146, 0.000599813, 0.00110407, 0.00200729, 0.00395649, 0.00786910, 0.0156436, 0.0295274, 0.0483376, 0.0949565}
   - The first orbits seem to be approaching 0 (but rather haphazardly), but the last orbit seems to move away from 0 at every step, roughly doubling each time.

   (c) Explain how to use the graphs of $y = x$, $y = f(x)$, and $y = f^2(x)$ below to locate (i) fixed points, (ii) periodic points of order exactly 2, and (iii) points x_0 such that $f(x_0)$ is fixed but x_0 is not fixed. (Do not do any computations.)



- For (i), the fixed points are the intersections between $y = f(x)$ and $y = x$, which are the yellow and green curves shown in the first plot.
- For (ii), a point will be periodic of order 2 whenever $y = f^2(x)$ intersects $y = x$ but not $y = f(x)$. On the third plot, these will be the points where the green and blue curves but not the yellow curve intersect.
- For (iii), the points such that $f(x_0)$ is fixed but x_0 is not fixed are the points where $y = f^2(x)$ intersects $y = f(x)$ but not $y = x$ will be an eventually fixed point. On the third plot, these will be the points where the yellow and blue curves but not the green curve intersect.
- The pictures suggest that there are infinitely many 2-cycles for this function as we approach $x = 0$, and also infinitely many points whose image is a fixed point. (In fact, there seem to be infinitely many of the latter near every zero of f.) Using some more careful analysis, one can prove both of these facts.

(d) Show that, in any open interval $(0, \epsilon)$ for any positive ϵ, there are infinitely many points such that $f(x) > x$, infinitely many points such that $f(x) = x$, and infinitely many points such that $f(x) < x$. Explain why this makes it impossible to characterize the orbit behavior near 0 as "attracting" or "repelling".

- The point is that $f(x) > x$, $f(x) = x$, and $f(x) < x$ will occur, respectively, when $\sin\left(\frac{1}{x}\right) > 0$, $\sin\left(\frac{1}{x}\right) = 0$, and $\sin\left(\frac{1}{x}\right) < 0$.
- So $f(x) > x$ on the intervals of the form $\left(\frac{1}{\pi + 2k\pi}, \frac{1}{2k\pi}\right)$ and $f(x) < x$ on the intervals of the form $\left(\frac{1}{2\pi + 2k\pi}, \frac{1}{\pi + 2k\pi}\right)$ for nonnegative integers k, and there are infinitely many intervals of each type in any open interval $(0, \epsilon)$. Likewise, there are infinitely many fixed points, since those have the form $\frac{1}{n\pi}$.
- For the last piece, the above tells us that in any interval around 0, there will be some points that f moves closer to 0 and others that f moves farther away. (In fact, it can be proven that there are always infinitely many points in any interval around 0 whose orbit will contain a value larger than 0.1, and infinitely many others whose orbit has limit 0.)