E. Dummit's Math 3543 ∼ Dynamics, Chaos, and Fractals, Spring 2025 ∼ Homework 2, due Wed Jan 22nd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Each of these functions has a neutral fixed point. Find it (you do not need to show it is unique), and then determine whether it is weakly attracting or weakly repelling for orbits on each side, making sure to include brief justification for the behavior:

   (a) $m(x) = x + x^5$.

   (b) $a(x) = x - x^4 + x^7$.

   (c) $t(x) = \sin(x)$.

   (d) $h(x) = e^{x/e}$.

   (e) $y(x) = \ln(1 - x)$.

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2. Let $p_c(x) = -x + x^2 + cx^3$, where c is a constant, and notice that 0 is a neutral fixed point. Determine (in terms of c) when 0 is weakly attracting and when it is weakly repelling.

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3. Let $p(x) = x^3 - ax$ for a parameter $a > 1$.

   (a) Find the three fixed points of p and classify them as attracting, repelling, or neutral (in terms of a).

   (b) Find a 2-cycle for p of the form $\{x_0, -x_0\}$ and classify it as attracting, repelling, or neutral (in terms of a).

   (c) For $a = 5/2$, it turns out that p has two other 2-cycles in addition to the one you found in part (b). Compute them explicitly, and classify them as attracting, repelling, or neutral.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose $\{x_0, x_1, x_2\}$ is a neutral 3-cycle for the function f and $f'(x_i) = 1$ for each $i = 0, 1, 2$.

   (a) If $g = f^3$, show that $g''(x_0) = g''(x_1) = g''(x_2) = f''(x_0) + f''(x_1) + f''(x_2)$. [Hint: Use the product and chain rules to compute $g''(x)$, then set $x = x_0$.]

   (b) Let $p(x) = 1 + x - 3x^2 - \frac{15}{4}x^3 + \frac{3}{2}x^4 + \frac{9}{4}x^5$. Show that 0 lies on a neutral 3-cycle for p, and classify the behavior of p^3 near 0 as weakly attracting or repelling on each side of 0. [Hint: Use (a).]

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5. Consider the two functions

$$\begin{aligned} s_1(x) &= -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{653}{1000}x^{10} \\ s_2(x) &= -x + x^2 - x^3 + \frac{3}{2}x^4 - \frac{5}{2}x^5 + \frac{19}{8}x^6 - \frac{139}{80}x^8 + \frac{652}{1000}x^{10} \end{aligned}$$

and observe that they both have a neutral fixed point at $x_0 = 0$.

(a) Numerically compute the first five elements in the orbits of 0.1 and 0.01 under both s_1 and s_2 to at least 30 decimal places.

(b) Based only on the orbits of 0.1 and 0.01, decide whether you think 0 is weakly attracting or weakly repelling for s_1 and s_2.

(c) Now compute the five elements in the orbit of 0.001 under s_1 and s_2 to 40 decimal places. Do the results agree with your guess from part (b)?

(d) Show that the neutral fixed point is weakly repelling for one of s_1, s_2 but weakly attracting for the other.

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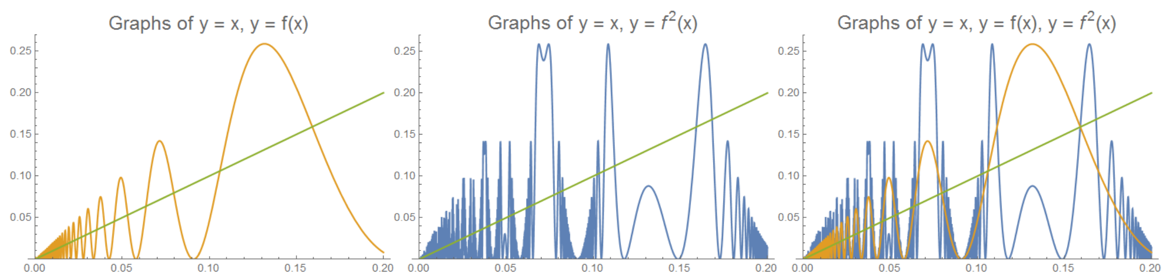
6. The goal of this problem is to explore a pathological example: a fixed point of a non-differentiable function. Let $f(x) = \begin{cases} x + x\sin\left(\dfrac{1}{x}\right) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$: note that f maps $[0, \infty) \to [0, \infty)$, and that f is continuous but not differentiable at $x = 0$.

(a) Find all the fixed points of f, and show that (except for $x = 0$) they are all repelling.

(b) Numerically compute the first 10 elements in the orbits of each of $x_0 = 0.1$, 0.01, 0.001, 0.0001, and 0.00015494157427205179. Are they being attracted to or repelled from zero in a consistent way?

(c) Explain how to use the graphs of $y = x$, $y = f(x)$, and $y = f^2(x)$ below to locate (i) fixed points, (ii) periodic points of order exactly 2, and (iii) points x_0 such that $f(x_0)$ is fixed but x_0 is not fixed. (Do not do any computations.)



(d) Show that, in any open interval $(0, \epsilon)$ for any positive ϵ, there are infinitely many points such that $f(x) > x$, infinitely many points such that $f(x) = x$, and infinitely many points such that $f(x) < x$. Explain why this makes it impossible to characterize the orbit behavior near 0 as "attracting" or "repelling".
