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1. Let $g(x) = x^2$.

   (a) Compute the first four points on the orbit of 1, on the orbit of -2, and on the orbit of $1/2$.

   - $\boxed{1, 1, 1, 1}$, $\boxed{-2, 4, 16, 256}$, $\boxed{\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}}$.

   (b) Compute $g^2(x)$, $g^3(x)$, and $g^4(x)$. What is the general formula for $g^n(x)$?

   - We have $g^2(x) = \boxed{x^4}$, $g^3(x) = \boxed{x^8}$, $g^4(x) = \boxed{x^{16}}$, and in general, $g^n(x) = \boxed{x^{2^n}}$.

   (c) Find all real numbers x that are eventually periodic points for g.

   - Using the formula from (b) we need $x^{2^n} = x$, which factors as $x(x^{2^n} - 1) = 0$.
   - Thus we get either $x = 0$ or $x^{2^n} = 1$ yielding $x = \pm 1$.
   - We have $0 \to 0 \to 0 \to \cdots$ so 0 is a periodic point of order 1, and also $1 \to 1 \to 1 \to \cdots$ so 1 is a periodic point also of order 1.
   - Finally, $-1 \to 1 \to 1 \to 1 \to \cdots$ so -1 is eventually periodic.

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2. For each function f and each point below, identify whether the point is (i) periodic, (ii) eventually periodic, or (iii) non-periodic.

   (a) $f(x) = x^2 - 2$, with points $x = \sqrt{2}$, $x = \sqrt{5}$, $x = -1$, $x = \frac{1+\sqrt{5}}{2}$.

   - We have $\sqrt{2} \to 0 \to -2 \to 2 \to 2 \to 2 \to \cdots$, so $\sqrt{2}$ is $\boxed{\text{eventually periodic}}$.
   - We have $\sqrt{5} \to 3 \to 7 \to 47 \to 2207 \to 4870847 \to \cdots$ and it is clear that the values will continue growing in absolute value, so $\sqrt{5}$ is $\boxed{\text{non-periodic}}$.
   - We have $-1 \to -1 \to -1 \to \cdots$, so -1 is $\boxed{\text{periodic}}$: indeed it is a fixed point.
   - We have $\frac{1+\sqrt{5}}{2} \to \frac{-1+\sqrt{5}}{2} \to \frac{-1-\sqrt{5}}{2} \to \frac{-1+\sqrt{5}}{2} \to \cdots$, so $\frac{1+\sqrt{5}}{2}$ is $\boxed{\text{eventually periodic}}$.

   (b) $f(x) = \frac{1}{3}(3 + 5x - 2x^3)$, with points $x = \frac{1}{2}\sqrt{10}$, $x = -3$, $x = 0$, and $x = 3$.

   - We have $\frac{1}{2}\sqrt{10} \to 1 \to 2 \to -1 \to 0 \to 1 \to \cdots$, so $\frac{1}{2}\sqrt{10}$ is $\boxed{\text{eventually periodic}}$.
   - We have $-3 \to 14 \to -1805 \to 3820487076 \to \cdots$ and it is clear that the values will continue growing in absolute value, so -3 is $\boxed{\text{non-periodic}}$.
   - We have $0 \to 1 \to 2 \to -1 \to 0 \to \cdots$, so 0 is $\boxed{\text{periodic}}$.
   - We have $3 \to -12 \to 1133 \to -969611202 \to \cdots$ and it is clear that the values will continue growing in absolute value, so 3 is $\boxed{\text{non-periodic}}$.

   (c) $f(x) = |2x - 2| - x$, with points $x = 5$, $x = 10$, $x = \frac{5}{3}$, $x = -\frac{1}{5}$.

   - We have $5 \to 3 \to 1 \to -1 \to 5 \to \cdots$, so 5 is $\boxed{\text{periodic}}$.
   - We have $10 \to 8 \to 6 \to 4 \to 2 \to 0 \to 2 \to \cdots$, so 2 is $\boxed{\text{eventually periodic}}$.
   - We have $\frac{5}{3} \to -\frac{1}{3} \to 3 \to 1 \to -1 \to 5 \to 3 \to \cdots$, so $\frac{5}{3}$ is $\boxed{\text{eventually periodic}}$.
   - We have $-\frac{1}{5} \to \frac{13}{5} \to \frac{3}{5} \to \frac{1}{5} \to \frac{7}{5} \to -\frac{3}{5} \to \frac{19}{5} \to \frac{9}{5} \to -\frac{1}{5} \to \cdots$, so $-\frac{1}{5}$ is $\boxed{\text{periodic}}$.

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3. Find all real fixed points (if there are any) for the following functions, and classify each of them as attracting, repelling, or neutral.

   (a) $a(x) = x^2 - 4x + 4$.

   - $x^4 - 4x + 4 = x$ gives $(x-1)(x-4) = 0$ so the fixed points are $x = \boxed{1,\ 4}$.
   - Since $a'(1) = -2$ and $a'(4) = 4$, they are $\boxed{\text{both repelling}}$.

   (b) $b(x) = x^2 + 4$.

   - $x^4 + 4 = x$ gives $x = \dfrac{1 \pm i\sqrt{15}}{2}$ so there are $\boxed{\text{no real fixed points}}$.

   (c) $c(x) = x^2 - \frac{1}{2}x + \frac{1}{2}$.

   - The fixed points are $x = \boxed{1, \frac{1}{2}}$. We have $c'(1) = \frac{3}{2}$ so $\boxed{1 \text{ is repelling}}$, and $c'(\frac{1}{2}) = \frac{1}{2}$ so $\boxed{\frac{1}{2} \text{ is attracting}}$.

   (d) $d(x) = x^3$.

   - $x^3 = x$ gives $x(x^2 - 1) = 0$ so $x = \boxed{-1,\ 0,\ 1}$.
   - We have $d'(-1) = 3$ so $\boxed{-1 \text{ is repelling}}$, $d'(0) = 0$ so $\boxed{0 \text{ is attracting}}$, and $d'(1) = 3$ so $\boxed{-1 \text{ is repelling}}$.

   (e) $e(x) = \dfrac{10}{x^2+1}$.

   - $\dfrac{10}{x^2+1} = x$ gives $x^3 + x - 10 = 0$, which factors as $(x-2)(x^2+2x+5) = 0$ and the quadratic has roots $x = -1 \pm 2i$. So the only real fixed point is $x = \boxed{2}$.
   - We have $e'(2) = -\frac{8}{5}$ so $\boxed{2 \text{ is repelling}}$.

   (f) $f(x) = \dfrac{2x^3}{3x^2-1}$.

   - Solving $x = \dfrac{2x^3}{3x^2-1}$ gives $x = \boxed{0, -1, 1}$. We also have $f'(x) = \dfrac{6(x^4 - x^2)}{(3x^2-1)^2}$, after simplification.
   - Then $f'(0) = 0$, $f'(1) = 0$, and $f'(-1) = 0$, so $\boxed{\text{each fixed point is attracting}}$.

   (g) $g(x) = x\cos(x)$.

   - $x\cos(x) = x$ factors as $x[\cos(x) - 1] = 0$, so $x = 0$ or $\cos(x) = 1$. The latter happens precisely when $x = 2\pi k$ for an integer k (which subsumes the case $x = 0$), so the fixed points are $x = \boxed{2\pi k \text{ for any integer } k}$.
   - We have $g'(x) = \cos(x) - x\sin(x)$, so $g'(2\pi k) = 1$ for all integers k. Thus, $\boxed{\text{each fixed point is neutral}}$.

   (h) $h(x) = |x|$.

   - $|x| = x$ is true whenever $\boxed{x \geq 0}$. Since $w'(x) = 1$ for $x > 0$ and $w'(0)$ is undefined, all of these fixed points are $\boxed{\text{neutral}}$.

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4. For each of the following functions $f(x)$, the point $x = 0$ lies in a periodic orbit. Classify this orbit as attracting, repelling, or neutral:

   (a) $f(x) = 1 - \frac{3}{2}x^2 - \frac{1}{2}x^3$.

   - We have $0 \to 1 \to -1 \to 0$ so the orbit has length 3. Since $f'(x) = -3x - \frac{3}{2}x^2$ we see $f'(0) = 0$, $f'(1) = -\frac{9}{2}$, and $f'(-1) = \frac{3}{2}$, so by the chain rule formula we see that $(f^3)'(0) = 0 \cdot (-\frac{9}{2}) \cdot \frac{3}{2} = 0$. Hence the orbit is $\boxed{\text{attracting}}$.

(b) $f(x) = 5 - x$.

- We have $0 \to 5 \to 0$ so the orbit has length 2. Since $f'(x) = -1$ we see $f'(0) = f'(5) = -1$ so by the chain rule formula we see that $(f^2)'(0) = (-1) \cdot (-1) = 1$. Hence the orbit is $\boxed{\text{neutral}}$.

(c) $f(x) = 2 + 2x - \frac{1}{4}x^2 - \frac{1}{4}x^3$.

- We have $0 \to 2 \to 3 \to -1 \to 0$ so the orbit has length 4. Since $f'(x) = 2 - \frac{1}{2}x - \frac{3}{4}x^2$ we see $f'(0) = 2$, $f'(2) = -2$, $f'(3) = -\frac{25}{4}$, $f'(-1) = \frac{7}{4}$, so by the chain rule formula we see that $(f^4)'(0) = 2 \cdot (-2) \cdot (-\frac{25}{4}) \cdot \frac{7}{4} = \frac{175}{4}$. Hence the orbit is $\boxed{\text{repelling}}$.

(d) $f(x) = \sqrt{2}(x^2 - x) + (1 - x)$.

- We have $0 \to 1 \to 0$ so the orbit has length 2. Since $f'(x) = \sqrt{2}(2x - 1) - 1$ we see $f'(0) = -\sqrt{2} - 1$ and $f'(1) = \sqrt{2} - 1$ so by the chain rule formula we see that $(f^2)'(0) = (-\sqrt{2} - 1)(\sqrt{2} - 1) = -1$. Hence the orbit is $\boxed{\text{neutral}}$.

(e) $f(x) = 1 + 0.1x + 2.1x^2 - 1.2x^3$.

- We have $0 \to 1 \to 2 \to 0$ so the orbit has length 3. Since $f'(x) = 0.1 + 4.2x - 3.6x^2$ we see $f'(0) = 0.1$, $f'(1) = 0.7$, and $f'(2) = -5.9$, so by the chain rule formula we see that $(f^3)'(0) = 0.1 \cdot 0.7 \cdot (-5.9) = -0.413$. Hence the orbit is $\boxed{\text{attracting}}$.

(f) $f(x) = -\frac{4}{\pi}\tan^{-1}(x + 1)$.

- We have $0 \to -1 \to 0$ so the orbit has length 2. Since $f'(x) = -\frac{4}{\pi} \cdot \frac{1}{1 + (x + 1)^2}$ we see $f'(0) = -\frac{2}{\pi}$, $f'(-1) = -\frac{4}{\pi}$, so by the chain rule formula we see that $(f^2)'(0) = \frac{8}{\pi^2} < 1$. Hence the orbit is $\boxed{\text{attracting}}$.

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5. Find the two fixed points and the unique 2-cycle for the function $p(x) = x^2 - 7$, and classify each of them as attracting, repelling, or neutral.

- Solving $p(x) = x$ gives $x = \boxed{\frac{1 \pm \sqrt{29}}{2}}$. Since $p'(x) = 2x$ we see that p' is equal to $1 \pm \sqrt{29}$ at the respective fixed points, so both of them are $\boxed{\text{repelling}}$.
- We can compute $\frac{p(p(x)) - x}{p(x) - x} = \frac{x^4 - 14x^2 - x + 42}{x^2 - x - 7} = x^2 + x - 6$, whose roots give the 2-cycle $\boxed{\{2, -3\}}$.
- We have $p'(2) = 4$ and $p'(-3) = -6$, so by the classification of cycles, since $4 \cdot (-6)$ has absolute value greater than 1, the cycle is $\boxed{\text{repelling}}$.

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6. Let $m(x) = \frac{1}{1 - x}$, defined for $x \neq 1$.

(a) Does $m(x)$ have any real fixed points? What about $m^2(x)$?

- For m, we would need $\frac{1}{1 - x} = x$, which is the same as $x^2 - x + 1 = 0$. This has no real solutions, so $\boxed{m \text{ has no real fixed points}}$.
- We have $m^2(x) = \frac{1}{1 - \frac{1}{1 - x}} = 1 - \frac{1}{x}$, so to get a fixed point we would need $1 - \frac{1}{x} = x$. But this is the same as $x^2 - x + 1 = 0$, so as above, $\boxed{m^2 \text{ has no real fixed points}}$.

(b) Find the first ten elements in the orbit of $x = 2$ under m.

- $\boxed{2, -1, \frac{1}{2}, 2, -1, \frac{1}{2}, 2, -1, \frac{1}{2}, -1}$. (It is a 3-cycle, as mandated by part (c).)

(c) Show that every value of x (except $x = 0$ and $x = 1$) lies on a 3-cycle under m.

- We have $m^3(x) = m^2(m(x)) = 1 - \dfrac{1}{1/(1-x)} = x$, provided that $x \neq 1$ and $x \neq 0$ (since we need both $m(x)$ and $m^2(x)$ to be defined).
- Since $m^3(x) = x$ for every x, that means every value of $x \neq 0, 1$ lies on a cycle of length at most 3. But by part (a) we know there are no fixed points for m, so there are no cycles of length less than 3. So every such point lies on a cycle of length exactly 3.

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7. Let $f(x) = e^x - \tan(x)$.

(a) Show that $f(x)$ has a fixed point in the interval $(1.0, 1.1)$.

- We use the Intermediate Value Theorem on $g(x) = f(x) - x = e^x - \tan(x) - x$, since this function is continuous.
- Because $g(1.0) = 0.1609$ and $g(1.1) = -0.0606$ (to four decimal places), the function g necessarily has a zero α in the interval $(1.0, 1.1)$.
- Then $g(\alpha) = 0$ says equivalently that $f(\alpha) = \alpha$, so that α is a fixed point of f, as desired.

(b) Show that $f(x)$ does not have a fixed point in the interval $(0.7, 0.8)$. [Hint: Show that $f(x) - x$ is positive on this interval.]

- On the interval $(0.7, 0.8)$ we have $f(x) - x = e^x - \tan(x) - x \geq e^{0.7} - \tan(0.8) - 0.8 = 0.1841$ to four decimal places.
- This means $f(x) - x$ is positive on the entire interval, so $f(x)$ cannot equal x there meaning that f cannot have a fixed point.

(c) Show that $f(x)$ has a 2-cycle $\{\alpha, \beta\}$ where $0.7 < \alpha < 0.8$.

- We use the Intermediate Value Theorem on $h(x) = f^2(x) - x$, since this function is continuous.
- Because $h(0.7) = 0.1571$ and $h(0.8) = -0.0347$ (to four decimal places), the function h necessarily has a zero α in the interval $(0.7, 0.8)$.
- Then $h(\alpha) = 0$ says equivalently that $f^2(\alpha) = \alpha$. By (b), this value α cannot be a fixed point of f, so α has order 2 and so $\{\alpha, f(\alpha)\}$ is a 2-cycle.

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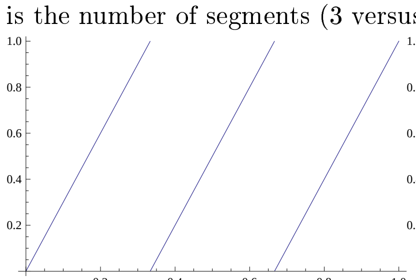
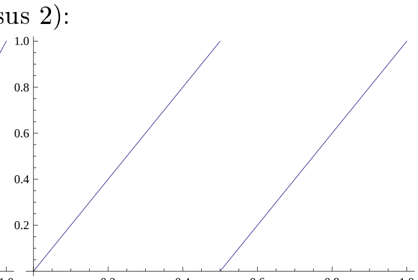
8. Let $f : [0, 1) \to [0, 1)$ be defined as $f(x) = \begin{cases} 3x & \text{if } 0 \leq x < 1/3 \\ 3x - 1 & \text{if } 1/3 \leq x < 2/3 \\ 3x - 2 & \text{if } 2/3 \leq x < 1 \end{cases}$. Observe that $f(x) = 3x$ modulo 1, which is also equivalent to saying that $f(x) = \{3x\}$ is the fractional part of $3x$.

(a) Sketch the graph of $y = f(x)$ and compare it to the doubling function $D(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1 \end{cases}$.

- The plot of $y = f(x)$ is on the left, while the doubling function is on the right. The only difference is the number of segments (3 versus 2):





(b) Find all the fixed points of f.

- If $0 \le x < \frac{1}{3}$ then $3x = x$ gives $x = 0$.
- If $\frac{1}{3} \le x < \frac{2}{3}$ then $3x - 1 = x$ gives $x = \frac{1}{2}$.
- If $\frac{2}{3} \le x < 1$ then $3x - 2 = x$ gives $x = 1$, which fails.
- So there are two fixed points: $x = \boxed{0,\ \frac{1}{2}}$.

(c) Describe the orbits of $x = \frac{1}{3}, \frac{1}{4}, \frac{1}{7}$, and $\frac{1}{45}$ under f.

- We have $\frac{1}{3} \to 0 \to 0 \to 0 \to 0 \to \cdots$, so the orbit settles at 0.
- Next, $\frac{1}{4} \to \frac{3}{4} \to \frac{1}{4} \to \frac{3}{4} \to \frac{1}{4} \to \cdots$, so the orbit is a 2-cycle.
- Third, $\frac{1}{7} \to \frac{3}{7} \to \frac{2}{7} \to \frac{6}{7} \to \frac{4}{7} \to \frac{5}{7} \to \frac{1}{7} \to \cdots$, so the orbit is a 6-cycle.
- Finally, $\frac{1}{45} \to \frac{1}{15} \to \frac{1}{5} \to \frac{3}{5} \to \frac{4}{5} \to \frac{2}{5} \to \frac{1}{5} \to \cdots$, so we see the orbit eventually settles in a 4-cycle.

(d) Suppose $x = \frac{p}{q}$ is a rational number. Show that x is either periodic or eventually periodic for f. [Hint: Consider the numerator and denominator of $f(x)$.]

- Suppose $x = \frac{p}{q}$. Note that $f(x)$ will be another rational number in $[0, 1)$ with denominator q, though possibly not in lowest terms.
- Since there are only q such numbers (namely, $\frac{0}{q}, \frac{1}{q}, \ldots, \frac{q-1}{q}$), we will eventually see that a value repeats, meaning that $\frac{p}{q}$ falls into a cycle.

(e) Suppose x is a periodic point of exact period k. Show that x must be a rational number and in fact that the denominator of x divides $3^k - 1$. [Hint: Use the fact that $f^n(x) - 3^n x$ is an integer.]

- As noted above, $f(x) = 3x$ modulo 1. Thus, $f^2(x) = 9x$ modulo 1, $f^3(x) = 27x$ modulo 1, and in general, $f^n(x) = 3^n x$ modulo 1. But by definition, this means $f^n(x) - 3^n x$ is an integer.
- Now suppose $f^n(x) = x$. By the above, we have $f^n(x) = 3^n x - k$ where k is an integer, so putting this together yields $x = 3^n x - k$ so $x = \frac{k}{3^n - 1}$.
- This expression is a rational number whose denominator in lowest terms must divide $3^n - 1$.

(f) Find five points that are periodic of exact order 3 for f.

- We will find all such points. By part (e) such a point must be a rational number of the form $\frac{p}{q}$ where q must divide $3^3 - 1 = 26$. Thus, $x = \frac{p}{26}$ for some integer p with $0 \le p \le 25$.
- Then $f(f(f(x))) = \frac{27p}{26}$ modulo 1, and it is not hard to see that this is equal to $\frac{p}{26} = x$. So $f(f(f(x))) = x$ for all such x.
- Finally, all such values will have period exactly 3, except for $p = 0$ and $p = 13$, which are the two fixed points from part (b).
- So the points of exact order 3 are the values $\boxed{\frac{p}{26} \text{ for integers } 1 \le p \le 25, \text{ except } p = 13}$.
