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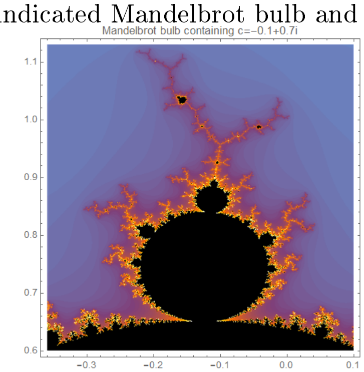
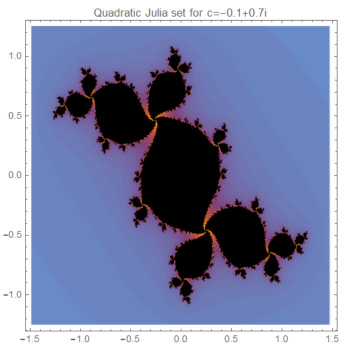
1. Each of the following values of c lies in a periodic bulb of the Mandelbrot set. For each value, (i) identify the p/q labeling of that bulb based on its location, (ii) check the result of the "antenna theorem" for the bulb, and (iii) check the result of the "lobe theorem" for the filled Julia set corresponding to that value of c. (Note that it may be difficult to visually judge the "smallest" antenna of the Mandelbrot bulb or the "largest" lobes of the Julia set, and at least one of the results will not agree with the prediction of the corresponding theorem!)

   (a) $c = -0.1 + 0.7i$.

   - Here is the indicated Mandelbrot bulb and the Julia set:

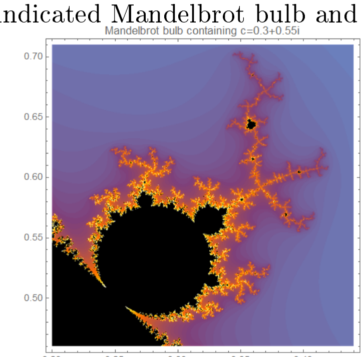


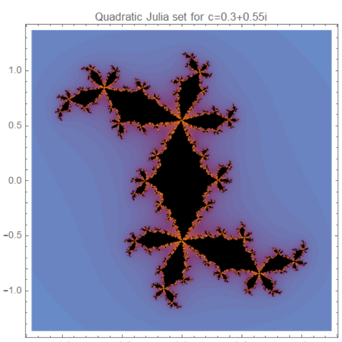


   - The antenna has 3 spokes, and the smallest one is the spoke immediately counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 1/3.
   - The filled Julia set has junction points with 3 lobes, and (looking at the junction point on the top right of the large central region) the first lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 1/3.
   - The theorems suggest that this value of c lies in the 1/3 bulb. By our theorem, the 1/3 bulb is tangent to the main cardioid at $c = \frac{1}{2}e^{2\pi i(1/3)} - \frac{1}{4}e^{4\pi i(1/3)} \approx -0.125 + 0.6495i$. A quick glance at the Mandelbrot set indicates that we have the correct bulb.

   (b) $c = 0.3 + 0.55i$.

   - Here is the indicated Mandelbrot bulb and the Julia set:

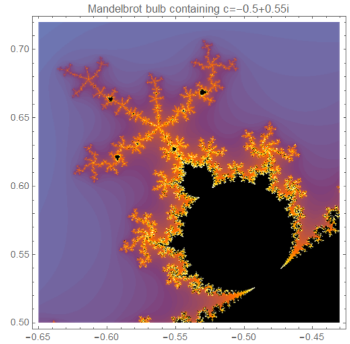


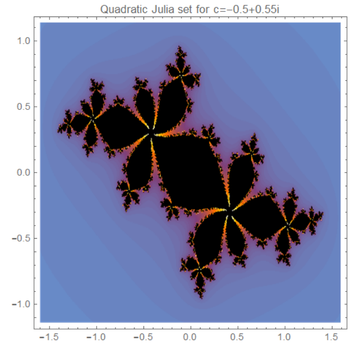


   - The antenna has 4 spokes, and the smallest one is the spoke immediately counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 1/4.
   - The filled Julia set has junction points with 4 lobes, and (looking at the junction point on the top of the large central region) the first lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 1/4.
   - The theorems suggest that this value of c lies in the 1/4 bulb. By our theorem, the 1/4 bulb is tangent to the main cardioid at $c = \frac{1}{2}e^{2\pi i(1/4)} - \frac{1}{4}e^{4\pi i(1/4)} \approx 0.25 + 0.5i$. A quick glance at the Mandelbrot set indicates that we have the correct bulb.

(c) $c = -0.5 + 0.55i$.

- Here is the indicated Mandelbrot bulb and the Julia set:

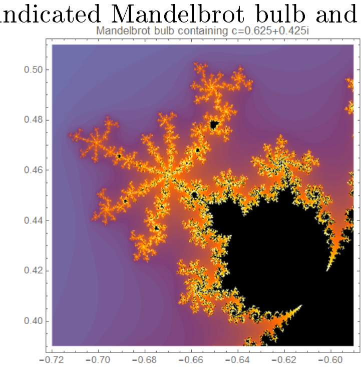


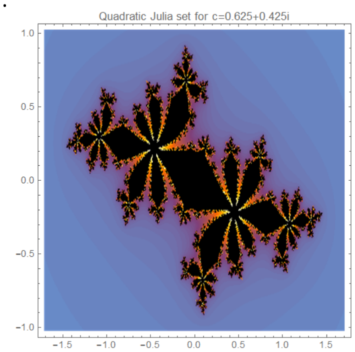


- The antenna has 5 spokes, and the smallest one is the spoke 2 counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 2/5, which is correct.
- The filled Julia set has junction points with 5 lobes, and (looking at the junction point on the top of the large central region) the second lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 2/5, which is correct.
- The theorems suggest that this value of c lies in the 2/5 bulb. By our theorem, the 2/5 bulb is tangent to the main cardioid at $c = \frac{1}{2}e^{2\pi i(2/5)} - \frac{1}{4}e^{4\pi i(2/5)} \approx -0.482 + 0.532i$. A quick glance at the Mandelbrot set indicates that we have the correct bulb.

(d) $c = -0.625 + 0.425i$.

- Here is the indicated Mandelbrot bulb and the Julia set:

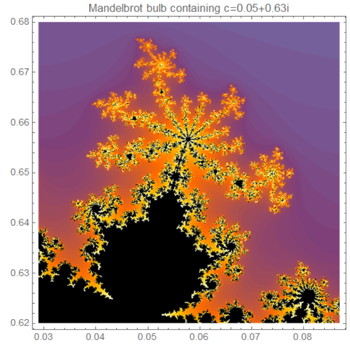


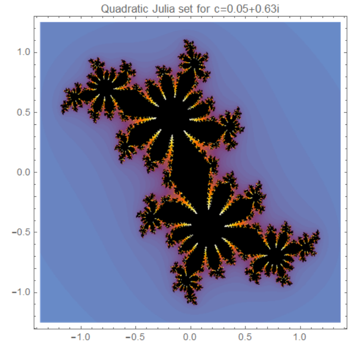


- The antenna has 7 spokes, and the smallest one is the spoke 3 counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 3/7.
- The filled Julia set has junction points with 7 lobes, and (looking at the junction point on the top of the large central region) the third lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 3/7.
- The theorems suggest that this value of c lies in the 3/7 bulb. By our theorem, the 3/7 bulb is tangent to the main cardioid at $c = \frac{1}{2}e^{2\pi i(3/7)} - \frac{1}{4}e^{4\pi i(3/7)} \approx -0.606 + 0.412i$. A quick glance at the Mandelbrot set indicates that we have the correct bulb.

(e) $c = 0.05 + 0.63i$.

- Here is the indicated Mandelbrot bulb and the Julia set:

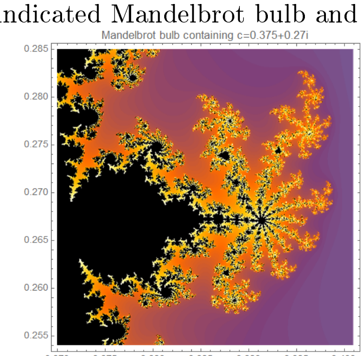


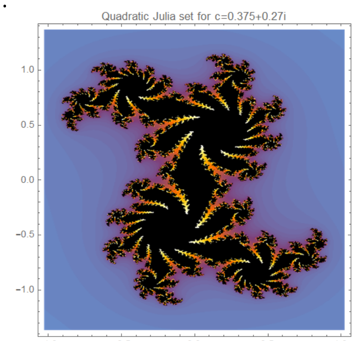


- The antenna has 10 spokes, and there are two small spokes that look to be about the same size: one is 3 counterclockwise from the starting spoke, and the other is 6 counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 3/10 or 6/10. (It is probably not 6/10 because the bulb labelings are always in lowest terms.)
- The filled Julia set has junction points with 10 lobes, and (looking at the junction point on the top of the large central region) the third lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 3/10, which is correct.
- The theorems suggest that this value of c lies in the 3/10 bulb. By our theorem, the 2/5 bulb is tangent to the main cardioid at $c = \frac{1}{2}e^{2\pi i(3/10)} - \frac{1}{4}e^{4\pi i(3/10)} \approx 0.048 + 0.622i$. A quick glance at the Mandelbrot set indicates that we have the correct bulb.

(f) $c = 0.375 + 0.27i$.

- Here is the indicated Mandelbrot bulb and the Julia set:





- The antenna has 11 spokes, and there are two small spokes that look to be about the same size: one is 4 counterclockwise from the starting spoke, and the other is 6 counterclockwise from the starting spoke. So the antenna theorem predicts the labeling should be 4/11 or 6/11.
- The filled Julia set has junction points with 11 lobes, and (looking at the junction point on the top of the central region) the second lobe clockwise is the second-largest. The Julia lobe theorem predicts the labeling should be 2/11.
- The theorems suggest a few different possible values of c. Searching for the appropriate tangency point to the main cardioid will quickly reveal that the correct value is 2/11, since $\frac{1}{2}e^{2\pi i(2/11)} - \frac{1}{4}e^{4\pi i(2/11)} \approx 0.371 + 0.266i$.
- If we look back at the antenna, we see that the second spoke is smaller than most of the others (it is about the third smallest of the 11 spokes) but it is definitely bigger than the 4th and 6th spokes!

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2. Choose three complex numbers c and plot (i) the Mandelbrot set near c, and (ii) the Julia set associated with $q_c(x) = x^2 + c$. In your response, please include the value of c and the Mandelbrot and Julia set plots.

   - As an open-ended problem, there are many possible responses.

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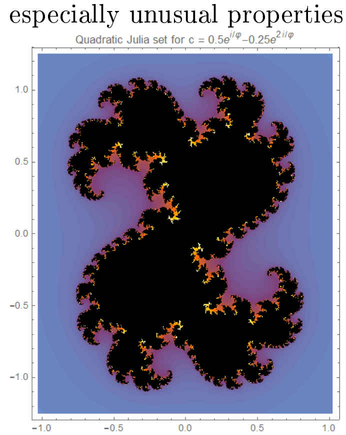
3. Observe in the plot of the 1/2 and 2/3 bulbs of the Mandelbrot set, there are various smaller bulbs progressing from the 1/2-bulb to the 2/3-bulb. Consider the sequence of bulbs B_1, B_2, B_3, B_4, ... with B_1 being the 1/2-bulb, B_2 being the 2/3-bulb, and for each $n \geq 2$, B_{n+1} is the largest bulb between B_n and B_{n-1}.

   (a) Find the labelings of the first five of these bulbs. Have you seen these numbers before?

   - From our results, the largest bulb between those with labelings a/b and c/d has labeling $(a+c)/(b+d)$.
   - So, iterating this starting with 1/2 and 2/3 produces the next five as $\boxed{3/5,\ 5/8,\ 8/13,\ 13/21,\ 21/34}$.
   - These ratios are in fact the famous Fibonacci numbers F_n, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
   - Indeed, it is easy to show by induction that the labeling of the nth bulb in the sequence (with 1/2 being the first) is F_n/F_{n+1} for each n: the first two are correct, and then if two consecutive bulbs are F_{n-1}/F_n and F_n/F_{n+1}, then the next is $(F_{n-1}+F_n)/(F_n+F_{n+1}) = F_{n+1}/F_{n+2}$.

   (b) It can be shown that the ratio p/q for this sequence of bulbs converges to $1/\varphi$ where $\varphi = (1+\sqrt{5})/2$ is the famous golden ratio. Plot the Julia set for the point $c = \frac{1}{2}e^{i/\varphi} - \frac{1}{4}e^{2i/\varphi}$ corresponding to this value on the main cardioid. Does it have any interesting features?

   - We can see some interesting self-similarity features, but this Julia set does not seem to have any especially unusual properties relative to other typical Julia sets for quadratic maps:



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4. There are many bulbs of the Mandelbrot set that are not attached directly to the main cardioid, but rather to another bulb. Each of these values of c lies in a "secondary bulb" attached to a "primary bulb" of the main cardioid: identify the p/q labeling of the primary bulb to which it is attached, and find the period inside the secondary bulb.

   (a) $c = -0.21 + 0.8i$.

   - By plotting the bulb in Mandel we can see that $c = -0.21 + 0.8i$ lies in a period-9 bulb attached to the 1/3-bulb, which has period 3. (Equivalently, we could compute 1000 or so iterates of the orbit of the critical point 0 and look at the period of the cycle it is attracted to. This is in fact exactly what Mandel does!)

   (b) $c = 0.13 - 0.63i$.

   - According to Mandel, $c = 0.13 - 0.63i$ lies in a period-14 bulb attached to the 5/7-bulb, which has period 7.

   (c) $c = 0.390 + 0.232i$.

   - According to Mandel, $c = 0.390 + 0.232i$ lies in a period-24 bulb attached to the 1/6-bulb, which has period 6.

   (d) $c = -0.547 - 0.557i$.

   - According to Mandel, $c = -0.547 - 0.557i$ lies in a period-15 bulb attached to the 3/5-bulb, which has period 5.

   (e) Is there any relation between the periods in the secondary bulb and the primary bulb?

   - In each case the period of the secondary bulb is a multiple of the period of the primary bulb. (In fact, this is true in great generality!)

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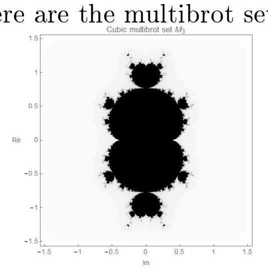
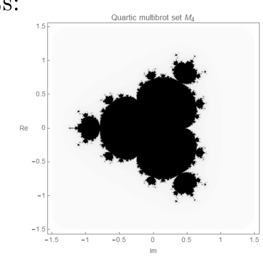
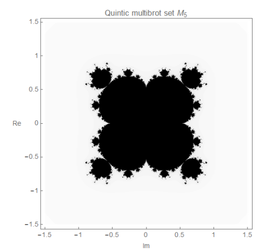
5. Let $p_{d,c}(z) = z^d + c$, where $d \geq 3$ is an integer, and observe that $p_{d,c}$ has a single critical point at $z = 0$. By the theorem on the structure of Julia sets for polynomial maps, we see that the Julia set for the map $p_{d,c} = z^d + c$ is either a totally disconnected Cantor-like set, or consists of a single connected component, according to whether the orbit of 0 escapes to ∞ or not. Define the multibrot set M_d to be the set of points $c \in \mathbb{C}$ for which the Julia set of $p_{d,c}$ is connected: equivalently, M_d is the set of points for which the orbit of 0 under $p_{d,c}$ remains bounded.

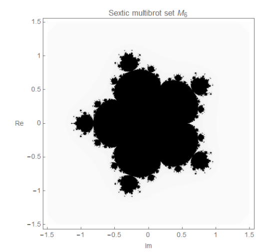
   (a) Plot the sets M_3, M_4, M_5, and M_6, and then compare them to one another and to the Mandelbrot set M. [In Mandel, you can plot these sets using the "Function" menu. Typing "q" will open a menu to change the value of d.]

   - Here are the multibrot sets:









   - Each of the sets has a vaguely similar shape: a central large lobed region (with an attracting fixed point), with smaller bulbs attached to the boundary having attracting cycles of larger periods.
   - They have the same sort of fine structure as the Mandelbrot set, with each bulb containing various antennae and smaller bulbs attached to them. In general, the sets M_d possess more symmetries than the Mandelbrot set.

   (b) Show that if $|z| > \max(|c|, 2^{1/(d-1)})$, then there is a $\lambda > 1$ such that $|p_{d,c}(z)| \geq \lambda |z|$. Conclude that for any such z, the orbit of z under $p_{d,c}$ escapes to ∞. [Hint: Modify the proof of the escape criterion for quadratic maps.]

   - By the assumption that $|z|^{d-1} > 2$ there is a $\lambda > 1$ such that $|z|^{d-1} - 1 > \lambda$.
   - Then by the triangle inequality, we have $|p_{c,d}(z)| \geq |z|^d - |c| \geq |z|^d - |z| = |z|(|z|^{d-1} - 1) \geq \lambda |z|$.
   - Since $\lambda > 1$ this implies $|p_{c,d}(z)| \geq \lambda |z| > |z| > \max(|c|, 2^{1/(d-1)})$.
   - Thus, $p_{c,d}(z) > \max(|c|, 2^{1/(d-1)})$, so so we may iteratively apply the argument to see that $\left|p_{c,d}^k(z)\right| \geq \lambda^k |z|$, and since $\lambda > 1$ and $|z| > 2^{1/(d-1)}$ we conclude that $\left|p_{c,d}^k(z)\right| \to \infty$ as $k \to \infty$.

   (c) Show that every point in the multibrot set M_d lies within or on the circle of radius $2^{1/(d-1)}$ centered at the origin. [Hint: Consider the second iterate of 0 and use part (b).]

   - Suppose $|c| > 2^{1/(d-1)}$: then $q_c^2(0) = c^d + c$ has absolute value at least $\left|c^d\right| - |c| \geq |c| \cdot (|c|^{d-1} - 1) > |c|$ and by hypothesis $|c| = \max(|c|, 2^{1/(d-1)})$.
   - So by the escape criterion of part (b), we see that the orbit of 0 will escape to ∞.

   (d) Show that $p_{d,c}(z)$ is conjugate to $p_{d,\omega c}(z)$, where $\omega = e^{2\pi i/(d-1)}$.

   - Let $h(z) = \omega z$: clearly, h is a homeomorphism.
   - Also, $h(p_{d,c}(z)) = \omega z^d + \omega c = (\omega z)^d + \omega c = p_{d,\omega c}(h(z))$, so h is the required conjugation.

   (e) Show that the multibrot set M_d for any $d \geq 3$ is invariant under rotation by an angle of $2\pi/(d-1)$ radians about the origin. [Hint: Use part (d).]

   - By part (d), we see that $p_{d,c}$ is conjugate to $p_{d,\omega c}$. So in particular, the orbit of z will escape to ∞ under $p_{d,c}$ if and only if ωz escapes under $p_{d,\omega c}$.
   - Set $z = 0$: then the orbit of 0 will escape under $p_{d,c}$ if and only if the orbit of 0 will escape under $p_{d,\omega c}$.
   - Equivalently, c lies in the multibrot set M_d if and only if ωc does.
   - Since this holds for every c, and multiplication by ω is equivalent to rotating the plane by the angle $2\pi/(d-1)$, we conclude that the multibrot set M_d is invariant under rotation by $2\pi/(d-1)$.

(f) Prove that $p_{d,c}(z)$ has an attracting or neutral fixed point if and only if $c = re^{it} - r^d e^{idt}$ for some $0 \le t \le 2\pi$ and some real number r with $0 \le r \le d^{-1/(d-1)}$. [Hint: Suppose $\alpha = re^{it}$ is the fixed point where $r \ge 0$ is real. Find the condition on r and then solve for c.]

- Suppose $\alpha = re^{it}$ is the fixed point. For α to be attracting or neutral is equivalent to saying that $p_{d,c}(\alpha) = \alpha$ and $\left|p'_{d,c}(\alpha)\right| \le 1$.
- Since $p'_{d,c}(\alpha) = d\alpha^{d-1}$ we see $\left|p'_{d,c}(\alpha)\right| = \left|d\alpha^{d-1}\right| = d\,|\alpha|^{d-1} = dr^{d-1}$, so with $r \ge 0$ the second condition becomes simply $dr^{d-1} \le 1$ so that $r \le d^{-1/(d-1)}$.
- Then the condition $p_{d,c}(\alpha) = \alpha$ becomes $\alpha^d + c = \alpha$ and so $c = \alpha - \alpha^d = re^{it} - r^d e^{idt}$ for some r with $0 \le r \le d^{-1/(d-1)}$. Conversely, reversing all of the calculations above shows that for such c, the point $\alpha = re^{it}$ is in fact attracting or neutral so $p_{d,c}$ does have an attracting or neutral fixed point as desired.

(g) Conclude that the multibrot set M_d contains a "central lobe" consisting of the region inside the curve $z = d^{-1/(d-1)}(e^{it} - d^{-1}e^{idt})$ for $0 \le t \le 2\pi$. Plot the central lobe together with the multibrot set for $d = 3$ and $d = 4$.

- By a theorem from in class, any attracting cycle will attract a critical point of $p_{d,c}$, so since $p_{d,c}$ has only the critical point $z = 0$, if there is an attracting cycle then it will attract the critical orbit hence the critical orbit will be bounded.
- Thus, in particular, if $p_{d,c}$ has an attracting fixed point, the critical orbit will be bounded, and so the corresponding c will lie in the multibrot set M_d.
- By (f), for each z on the curve parametrized by $z = d^{-1/(d-1)}(e^{it} - d^{-1}e^{idt})$ for $0 \le t \le 2\pi$, the value $\alpha = d^{-1/(d-1)}e^{it}$ is a neutral fixed point, and for all z inside that curve, namely $z = re^{it} - r^d e^{idt}$ for some $0 \le r < d^{-1/(d-1)}$, the point $\alpha = re^{it}$ is an attracting fixed point and therefore c lies inside the set M_d.
- The central lobes and their boundary curves can be parametrized in xy-coordinates by taking real and imaginary parts: for instance the boundary curve is given by $x = d^{-1/(d-1)}(\cos t - d^{-1}\cos dt)$, $y = d^{-1/(d-1)}(\sin t - d^{-1}\sin dt)$, for $0 \le t \le 2\pi$.
- Here are the plots for $d = 3$ and $d = 4$: