E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 11, due Wed Apr 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Each of the following values of c lies in a periodic bulb of the Mandelbrot set. For each value,
   (i) identify the p/q labeling of that bulb based on its location,
   (ii) check the result of the "antenna theorem" for the bulb, and
   (iii) check the result of the "lobe theorem" for the filled Julia set corresponding to that value of c.
   (Note that it may be difficult to visually judge the "smallest" antenna of the Mandelbrot bulb or the "largest" lobes of the Julia set, and at least one of the results will not agree with the prediction of the corresponding theorem!)

   (a) $c = -0.1 + 0.7i$.
   (b) $c = 0.3 + 0.55i$.
   (c) $c = -0.5 + 0.55i$.
   (d) $c = -0.625 + 0.425i$.
   (e) $c = 0.05 + 0.63i$.
   (f) $c = 0.375 + 0.27i$.

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2. Choose three complex numbers c and plot (i) the Mandelbrot set near c, and (ii) the Julia set associated with $q_c(x) = x^2 + c$. In your response, please include the value of c and the Mandelbrot and Julia set plots.

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3. Observe in the plot of the 1/2 and 2/3 bulbs of the Mandelbrot set, there are various smaller bulbs progressing from the 1/2-bulb to the 2/3-bulb. Consider the sequence of bulbs B_1, B_2, B_3, B_4, ... with B_1 being the 1/2-bulb, B_2 being the 2/3-bulb, and for each $n \geq 2$, B_{n+1} is the largest bulb between B_n and B_{n-1}.

   (a) Find the labelings of the first five of these bulbs. Have you seen these numbers before?
   (b) It can be shown that the ratio p/q for this sequence of bulbs converges to $1/\varphi$ where $\varphi = (1 + \sqrt{5})/2$ is the famous golden ratio. Plot the Julia set for the point $c = \frac{1}{2}e^{i/\varphi} - \frac{1}{4}e^{2i/\varphi}$ corresponding to this value on the main cardioid. Does it have any interesting features?

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4. There are many bulbs of the Mandelbrot set that are not attached directly to the main cardioid, but rather to another bulb. Each of these values of c lies in a "secondary bulb" attached to a "primary bulb" of the main cardioid: identify the p/q labeling of the primary bulb to which it is attached, and find the period inside the secondary bulb.

   (a) $c = -0.21 + 0.8i$.
   (b) $c = 0.13 - 0.63i$.
   (c) $c = 0.390 + 0.232i$.
   (d) $c = -0.547 - 0.557i$.
   (e) Is there any relation between the periods in the secondary bulb and the primary bulb?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Let $p_{d,c}(z) = z^d + c$, where $d \geq 3$ is an integer, and observe that $p_{d,c}$ has a single critical point at $z = 0$. By the theorem on the structure of Julia sets for polynomial maps, we see that the Julia set for the map $p_{d,c} = z^d + c$ is either a totally disconnected Cantor-like set, or consists of a single connected component, according to whether the orbit of 0 escapes to ∞ or not. Define the multibrot set M_d to be the set of points $c \in \mathbb{C}$ for which the Julia set of $p_{d,c}$ is connected: equivalently, M_d is the set of points for which the orbit of 0 under $p_{d,c}$ remains bounded.

   (a) Plot the sets M_3, M_4, M_5, and M_6, and then compare them to one another and to the Mandelbrot set M. [In Mandel, you can plot these sets using the "Function" menu. Typing "q" will open a menu to change the value of d.]

   (b) Show that if $|z| > \max(|c|, 2^{1/(d-1)})$, then there is a $\lambda > 1$ such that $|p_{d,c}(z)| \geq \lambda |z|$. Conclude that for any such z, the orbit of z under $p_{d,c}$ escapes to ∞. [Hint: Modify the proof of the escape criterion for quadratic maps.]

   (c) Show that every point in the multibrot set M_d lies within or on the circle of radius $2^{1/(d-1)}$ centered at the origin. [Hint: Consider the second iterate of 0 and use part (b).]

   (d) Show that $p_{d,c}(z)$ is conjugate to $p_{d,\omega c}(z)$, where $\omega = e^{2\pi i/(d-1)}$.

   (e) Show that the multibrot set M_d for any $d \geq 3$ is invariant under rotation by an angle of $2\pi/(d-1)$ radians about the origin. [Hint: Use part (d).]

   (f) Prove that $p_{d,c}(z)$ has an attracting or neutral fixed point if and only if $c = re^{it} - r^d e^{idt}$ for some $0 \leq t \leq 2\pi$ and some real number r with $0 \leq r \leq d^{-1/(d-1)}$. [Hint: Suppose $\alpha = re^{it}$ is the fixed point where $r \geq 0$ is real. Find the condition on r and then solve for c.]

   (g) Conclude that the multibrot set M_d contains a "central lobe" consisting of the region inside the curve $z = d^{-1/(d-1)}(e^{it} - d^{-1}e^{idt})$ for $0 \leq t \leq 2\pi$. Plot the central lobe together with the multibrot set for $d = 3$ and $d = 4$.
